SCHRAMM-LOEWNER EVOLUTIONS, LENT 2019, EXAMPLE SHEET 1

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Problem 1.

- Suppose that $f\colon \mathbb{D} \to \mathbb{D}$ is a conformal transformation (i.e., f is a conformal automorphism of $\mathbb{D}$). Use the Schwarz lemma to show that there exists $z \in \mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ so that
$$f(w) = \lambda \frac{z-w}{\overline{z}w-1}.$$
- Suppose that $f\colon \mathbb{H} \to \mathbb{H}$ is a conformal transformation (i.e., f is a conformal automorphism of $\mathbb{H}$). Show that there exists $a,b,c,d \in \mathbb{R}$ with $ad-bc=1$ so that
$$f(z) = \frac{az+b}{cz+d}.$$
Deduce that if f fixes 0 and ∞ then there exists $a>0$ so that $f(z)=az$.

Problem 2.

- Using the conformal invariance of Brownian motion, show that the hitting density (with respect to Lebesgue measure) for a complex Brownian motion starting from $z \in \mathbb{D}$ on the unit circle is given by
$$p(z, e^{i\theta}) = \frac{1}{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2} \quad \text{for} \quad \theta \in [0,2\pi).$$
You may assume that the hitting density is given by the uniform distribution on $\partial\mathbb{D}$ when $z=0$.
- Using the conformal invariance of Brownian motion, show that the hitting density (with respect to Lebesgue measure) for a complex Brownian motion starting from $z \in \mathbb{H}$ on the real line $\partial\mathbb{H}$ is given by
$$p(z,u) = \frac{1}{\pi}\frac{y}{(x-u)^2+y^2} \quad \text{where} \quad z = x+iy, \quad u \in \partial\mathbb{H}.$$
(Note that $p(i,\cdot)$ is the Cauchy distribution on $\mathbb{R}$.)

Problem 3.

- Show that $f(z) = z + 1/z$ is a conformal transformation from $\mathbb{H} \setminus \overline{\mathbb{D}}$ to $\mathbb{H}$.
- Using the conformal invariance of Brownian motion, show that the density $p(z,e^{i\theta})$, $\theta \in [0,\pi]$, for the first exit distribution (with respect to Lebesgue measure) of a complex Brownian motion on $\mathbb{H} \cap \partial\mathbb{D}$ starting from $z \in \mathbb{H} \setminus \overline{\mathbb{D}}$ satisfies:
$$p(z,e^{i\theta}) = \frac{2}{\pi}\frac{\mathrm{Im}(z)}{|z|^2}\sin(\theta)\left(1 + O(|z|^{-1})\right) \quad \text{as} \quad z \to \infty.$$

Problem 4. Using the previous problem, show that if $A \in \mathcal{Q}$ with $A \subseteq \overline{\mathbb{D}} \cap \mathbb{H}$ then
$$\mathrm{hcap}(A) = \frac{2}{\pi}\int_0^\pi \mathbb{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)]\sin(\theta)d\theta$$

where τ is the first time that a complex Brownian motion B exits $\mathbb{H} \setminus A$ and $\mathbb{E}_z$ denotes the expectation with respect to the law under which B starts from z.

Problem 5. (Schwarz reflection for harmonic functions) Suppose that $u\colon \overline{\mathbb{H} \cap \mathbb{D}} \to \mathbb{R}$ is harmonic in $\mathbb{H} \cap \mathbb{D}$, continuous in $\overline{\mathbb{H} \cap \mathbb{D}}$, and vanishes on $[-1,1]$. Show that u extends to a harmonic function on $\mathbb{D}$ by odd reflection, i.e., by taking $u(\overline{z}) = -u(z)$.

Problem 6. Suppose that D is a domain in $\mathbb{C}$ and f is holomorphic and non-zero on D. Show that $\log|f|$ is harmonic.

Problem 7.

- Consider the rectangle $A_r = [-r, r] \times (0, 1]$ in $\mathbb{H}$. Show that there exists a constant $c > 0$ such that $\mathrm{hcap}(A_r) \leq cr$ for all $r \geq 1$.
- Find a sequence of compact $\mathbb{H}$-hulls (A_n) such that $\mathrm{diam}(A_n) \to \infty$ but $\mathrm{hcap}(A_n) \to 0$.

Problem 8. Suppose that u is a harmonic function on a domain $D \subseteq \mathbb{C}$. Show that for each $n \in \mathbb{N} = \{1, 2, \ldots\}$ there exists a constant $c_n > 0$ such that for all $j, k \in \mathbb{N}_0 = \{0, 1, \ldots\}$ with $j + k = n$ and $z = x + iy \in D$ we have that

$$\left|\partial_x^j \partial_y^k u(z)\right| \leq \frac{c_n}{\mathrm{dist}(z, \partial D)^n} \|u\|_\infty.$$

Hint: use the first part of Problem 2.

Problem 9. Suppose that $A \in \mathcal{Q}$ with $\mathrm{rad}(A) = \sup\{|z| : z \in A\} \leq 1$. Show that

$$x \leq g_A(x) \leq x + \frac{1}{x} \quad \text{for all} \quad x > 1$$
$$x + \frac{1}{x} \leq g_A(x) \leq x \quad \text{for all} \quad x < -1.$$

Show also that for all $A \in \mathcal{Q}$ and $A \in \mathbb{H} \setminus A$ we have that $|g_A(z) - z| \leq 3\mathrm{rad}(A)$. Hint: for $x > 1$, show that $g_A(x)$ is increasing in A and recall the first part of Problem 3.

Problem 10. Suppose that $A \in \mathcal{Q}$ is connected. Let B be a complex Brownian motion and let $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus A\}$. Show that there exists constants $c_1, c_2 > 0$ such that

$$c_1 \mathrm{diam}(A) \leq \lim_{y \to \infty} y \mathbb{P}_{iy}[B_\tau \in A] \leq c_2 \mathrm{diam}(A).$$

Problem 11. Suppose that $\gamma\colon [0, T] \to \overline{\mathbb{H}}$ is a simple curve (i.e., $s \neq t$ implies $\gamma(s) \neq \gamma(t)$) with $\gamma(0) = 0$ and $\gamma(t) \in \mathbb{H}$ for all $t \in (0, T]$. Show that $A_t = \gamma((0, t])$ for $t \in [0, T]$ is a family of locally growing compact $\mathbb{H}$-hulls. Show, moreover, that there exists a homeomorphism $\phi\colon [0, T] \to [0, \frac{1}{2}\mathrm{hcap}(A_T)]$ so that $\mathrm{hcap}(A_{\phi^{-1}(t)}) = 2t$ for all $t \in [0, \frac{1}{2}\mathrm{hcap}(A_T)]$. (This is the so-called capacity parameterization of γ.)

Problem 12. Suppose that $U\colon [0, T] \to \mathbb{R}$ is a continuous function. Let $g_t(z)$ solve the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Show for each $t \in [0, T]$ that g_t is a conformal transformation from its domain onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$ using the following steps.

- Show that $t \mapsto \mathrm{Im}(g_t(z))$ is decreasing in t, hence for each $z \in \mathbb{H}$, $t \mapsto g_t(z)$ is defined up until $\tau_z = \sup\{t \geq 0 : \mathrm{Im}(g_t(z)) > 0\}$. Conclude that $H_t = \{z : \tau_z > t\}$ is the domain of g_t.
- Show for each $t \in [0, T]$ that $z \mapsto g_t(z)$ is complex differentiable on H_t.

- Show for each $t \in [0,T]$ that $z \mapsto g_t(z)$ has an inverse defined on $\mathbb{H}$ by showing that $g_t(f_t(w)) = w$ for all $w \in \mathbb{H}$ where f_s for $s \in [0,t]$ solves the so-called *reverse chordal Loewner equation*
$$\partial_s f_s(w) = -\frac{2}{f_s(w) - U_{t-s}}, \quad f_0(w) = w.$$

Optional problems: Riemann mapping theorem

The purpose of this sequence of problems is to prove the Riemann mapping theorem.

Optional Problem 1. Prove the Harnack inequality: suppose that u is a positive harmonic function defined on a domain D. Then for each $K \subseteq D$ compact there exists a constant $M > 0$ (independent of u) such that
$$\frac{\sup_{z \in K} u(z)}{\inf_{z \in K} u(z)} \leq M.$$

Optional Problem 2. Deduce from Problem 1 that if $f, \widetilde{f}$ are conformal transformations $D \to \mathbb{D}$ taking z to 0 and with positive derivative at z, then $f = \widetilde{f}$.

Optional Problem 3. Suppose that D is a simply connected domain with $D \neq \mathbb{C}$. Suppose that $z \in D$. Show that there exists a unique conformal transformation $f \colon D \to \mathbb{D}$ with $f(z) = 0$ and $f'(z) > 0$ using the following steps.

- Let $\mathcal{C}$ be the collection of conformal transformations f from D into a subset of $\mathbb{D}$ with $f(z) = 0$ and $f'(z) > 0$. Deduce from the Schwarz lemma that if $f \in \mathcal{C}$ then $f'(z) \leq (\mathrm{dist}(z, \partial D))^{-1}$.
- Show that $\mathcal{C}$ is non-empty.
- Suppose that (f_n) is a sequence in $\mathcal{C}$ such that, for each $K \subseteq D$ compact, we have that $f_n|_K \to f|_K$ uniformly where f is conformal on D. Show that f is either constant or injective.
- Let $M = \sup\{f'(z) : z \in \mathcal{C}\}$. Let (f_n) be a sequence of functions in $\mathcal{C}$ with $f_n'(z)$ increasing to M. Explain why there exists a subsequence (f_{n_k}) of (f_n) which converges uniformly to a map $f \colon D \to \mathbb{D}$. (Hint: use Problem 7, the Harnack inequality, and the Arzela-Ascoli theorem.) Explain why $f'(z) = M$ and deduce from the previous part that f is injective.
- Show that f is surjective onto $\mathbb{D}$. (Hint: argue by contradiction that if f is not surjective then $f'(z) < M$.)