## SCHRAMM-LOEWNER EVOLUTIONS, LENT 2019, EXAMPLE SHEET 2

Please send corrections to jpmiller@statslab.cam.ac.uk

**Problem 1.** Suppose that $U_t = \sqrt{\kappa} B_t$ where $B$ is a standard Brownian motion and let $(g_t)$ solve

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

- (Markov property) Suppose that $\tau$ is a stopping time for $U$ which is almost surely finite and let $\widetilde{g}_t = g_{\tau+t}(g_\tau^{-1}(z + U_\tau)) - U_\tau$. Show that the maps $(\widetilde{g}_t)$ have the same distribution as the maps $(g_t)$.
- (Scale invariance) Fix $r > 0$ and let $\widetilde{g}_t(z) = r g_{t/r^2}(z/r)$. Show that the maps $(\widetilde{g}_t)$ have the same distribution as the maps $(g_t)$.

Suppose that $D$ is a simply connected domain, $x, y \in \partial D$ are distinct, and $\varphi \colon \mathbb{H} \to D$ is a conformal transformation with $\varphi(0) = x$ and $\varphi(\infty) = y$. Explain why the definition of $\mathrm{SLE}_\kappa$ given by $\varphi(\gamma)$ where $\gamma$ is an $\mathrm{SLE}_\kappa$ in $\mathbb{H}$ from 0 to $\infty$ is well-defined.

**Problem 2.**

- Suppose that $B$ is a standard Brownian motion and $a < 0$. Show that $\sup_{t \geq 0}(B_t + at) < \infty$ almost surely.
- Suppose that $(g_t)$ is the family of conformal maps which solve the Loewner equation with driving function $U_t = \sqrt{\kappa} B_t$ and, for each $x \in \mathbb{R}$, let $V_t^x = g_t(x) - U_t$ and $\tau_x = \inf\{t \geq 0 : V_t^x = 0\}$. For each $0 < x < y$, let $g(x,y) = \mathbb{P}[\tau_x = \tau_y]$. Show that if $g(1, 1+\epsilon/2) > 0$ for all $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$ then $g(x,y) > 0$ for all $0 < x < y$.

**Problem 3.** Fix $T > 0$ and let $D \subseteq \mathbb{H}$ be a simply connected domain. Suppose that $(A_t)_{t \in [0,T]}$ is a non-decreasing family of compact $\mathbb{H}$-hulls which are locally growing with $A_0 = \emptyset$, $\mathrm{hcap}(A_t) = 2t$ for all $t \in [0,T]$, and $A_T \subseteq D$. Let $\psi \colon D \to \mathbb{H}$ be a conformal transformation which is bounded on bounded sets. Show that the family of compact $\mathbb{H}$-hulls $\widetilde{A}_t = \psi(A_t)$ for $t \in [0,T]$ is locally growing with $\widetilde{A}_0 = \emptyset$ and with

$$\mathrm{hcap}(\widetilde{A}_t) = \int_0^t 2(\psi_s'(U_s))^2 ds \quad \text{where} \quad \psi_t = \widetilde{g}_t \circ \psi \circ g_t^{-1} \quad \text{for each} \quad t \in [0,T]$$

and $\widetilde{g}_t$ is the unique conformal transformation $\mathbb{H} \setminus \widetilde{A}_t \to \mathbb{H}$ with $\widetilde{g}_t(z) - z \to 0$ as $z \to \infty$.

**Problem 4.** In the setting of the previous problem, show that

$$\partial_t \psi_t(U_t) = \lim_{z \to U_t} \partial_t \psi_t(z) = -3\psi_t''(U_t).$$

**Problem 5.** Suppose that $(A_t)$ is a non-decreasing family of $\mathbb{H}$-hulls which are locally growing and with $A_0 = \emptyset$. For each $t \geq 0$, let $a(t) = \mathrm{hcap}(A_t)$ and assume that $a(t)$ is $C^1$. For each $t \geq 0$, let $g_t$ be the unique conformal transformation which takes $\mathbb{H} \setminus A_t$ to $\mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$. Show that the conformal maps $(g_t)$ satisfy the ODE:

$$\partial_t g_t(z) = \frac{\partial_t a(t)}{g_t(z) - U_t}, \quad g_0(z) = z$$

for some continuous, real-valued function $U_t$. (Hint: perform a time-change so that the hulls are parameterized by capacity, apply Loewner's theorem as proved in class, and then invert the time change.)

**Problem 6.** Suppose that $B$ is a standard Brownian motion starting from $B_0 = x > 0$. For each $a \in \mathbb{R}$, let $\tau_a = \inf\{t \geq 0 : B_t = a\}$.

- For $a < x < b$, explain why $\mathbb{P}[\tau_a < \tau_b] = (b-x)/(b-a)$.
- Using the Girsanov theorem, explain why the law of $B$ weighted by $B_{\tau_0 \wedge \tau_b}$ is equal to that of a $\mathrm{BES}^3$ process stopped upon hitting $b$. That is, if $\mathbb{P}$ denotes the law of $B$ and we define the law $\widetilde{\mathbb{P}}$ using the Radon-Nikodym derivative
$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{B_{\tau_0 \wedge \tau_b}}{\mathbb{E}[B_{\tau_0 \wedge \tau_b}]}$$
then the law of $B$ under $\widetilde{\mathbb{P}}$ is that of a $\mathrm{BES}^3$ process stopped upon hitting $b$.
- Explain why a standard Brownian motion conditioned to be non-negative is a $\mathrm{BES}^3$ process.
- More generally, explain why a $\mathrm{BES}^d$ process with $d < 2$ conditioned to be non-negative is a $\mathrm{BES}^{4-d}$ process.

**Problem 7.** Suppose that $(g_t)$ is the family of conformal maps associated with an $\mathrm{SLE}_\kappa$ with driving function $U_t$, i.e., $U_t = \sqrt{\kappa} B_t$ where $B$ is a standard Brownian motion. Fix $z \in \mathbb{H}$ and let $z_t = x_t + i y_t = g_t(z)$. Assume that $\rho \in \mathbb{R}$ is fixed. Use Itô's formula to show that
$$M_t = |g_t'(z)|^{(8-2\kappa+\rho)\rho/(8\kappa)} y_t^{\rho^2/8\kappa} |U_t - z_t|^{\rho/\kappa}$$
is a continuous local martingale. (Hint: let
$$Z_t = \frac{(8-2\kappa+\rho)\rho}{8\kappa} \log g_t'(z) + \frac{\rho^2}{8\kappa} \log y_t + \frac{\rho}{\kappa} \log(U_t - z_t),$$
compute $dZ_t$ using Itô's formula, take its real part, and exponentiate.)

**Problem 8.** Assume that we have the setup of Problem 7. Let $\Upsilon_t = y_t / |g_t'(z)|$.

- Explain why $\Upsilon_t$ is proportional to $\mathrm{dist}(z, \gamma([0,t]) \cup \partial\mathbb{H})$. More precisely, explain why
$$\frac{1}{4} \leq \frac{\Upsilon_t}{\mathrm{dist}(z, \gamma([0,t]) \cup \partial\mathbb{H})} \leq 4.$$
- Let $S_t = \sin(\arg(z_t - U_t))$. Explain why
$$M_t = |g_t'(z)|^{(8-\kappa+\rho)\rho/(4\kappa)} \Upsilon_t^{\rho(\rho+8)/(8\kappa)} S_t^{-\rho/\kappa}.$$
- By considering the above martingale with the special choice $\rho = \kappa - 8$, show that if $\kappa > 8$ then the $\mathrm{SLE}_\kappa$ curve $\gamma$ almost surely hits $z$. Conclude that $\gamma$ fills all of $\mathbb{H}$. (Hint: recall that we showed in class that $\gamma$ fills $\partial\mathbb{H}$. Deduce from this and the conformal Markov property that $\gamma$ cannot separate $z$ from $\infty$ without hitting it. Consider the behavior of $S_t$ when $\gamma$ is hitting a point on $\partial\mathbb{H}$ with either very large positive or negative values.)

**Problem 9.** In the context of Problem 4, show that
$$\partial_t \psi_t'(U_t) = \lim_{z \to U_t} \partial_t \psi_t'(z) = \frac{\psi_t''(U_t)^2}{2\psi_t'(U_t)} - \frac{4}{3}\psi_t'''(U_t).$$

**Problem 10.** Prove that the Dirichlet inner product is conformally invariant. That is, show that if $f, g \in C_0^\infty(D)$ and $\varphi\colon D \to \widetilde{D}$ is a conformal transformation, then

$$(f, g)_\nabla = (f \circ \varphi^{-1}, g \circ \varphi^{-1})_\nabla.$$

(Hint: use the change of variables formula and the Cauchy-Riemann equations.)

**Problem 11.** Suppose that $f \in H_0^1(D)$ with $\Delta f = 0$ in $U$ in the *distributional sense*: if $g \in C_0^\infty(U)$, then $(f, \Delta g) = 0$ where $(\cdot,\cdot)$ denotes the $L^2$ inner product. Show that $f|_U$ is $C^\infty$ in $U$ and $\Delta f = 0$ in $U$ in (the usual sense) using the following steps.

- Let $\phi$ be a radially symmetric $C_0^\infty$ bump function supported in $\mathbb{D}$. In other words, $\phi(x) \geq 0$ for all $x$, $\phi(x)$ depends only on $|x|$, $\phi(x) = 0$ for $|x| \geq 1$, and $\int \phi = 1$. For each $\epsilon > 0$, let
$$f_\epsilon(x) = \epsilon^{-2} \int f(y) \phi\left(\frac{x-y}{\epsilon}\right) dy.$$
Explain why $f_\epsilon$ is $C^\infty$ in $U_\epsilon = \{z \in U : \mathrm{dist}(z, \partial U) > \epsilon\}$.
- Fix $\delta > 0$ and let $x \in U_\delta$. Explain why $f_\epsilon(x)$ does not depend on the value of $\epsilon$ for $\epsilon \in (0, \delta)$. (Hint: compute the derivative of $f_\epsilon(x)$ respect to $\epsilon$, recall the form of $\Delta$ when expressed in polar coordinates, and consider the radially symmetric function $\psi(r) = \int r\phi(r)dr$.)
- Conclude that if $g \in C_0^\infty(U)$, then the value of $(f_\epsilon, g)$ does not depend on $\epsilon$ for sufficiently small values of $\epsilon$.
- Explain why the previous parts imply that $f$ is $C^\infty$ in $U$ and $\Delta f = 0$ in $U$ (in the usual sense).

**Bonus Problem.** Fill in the missing details to the proof of Theorem 11.3 from the lecture notes by proving the following.

- Suppose that $\gamma$ is an $\mathrm{SLE}_{8/3}$ in $\mathbb{H}$ from 0 to $\infty$. Suppose that for every $A \in \mathcal{Q}_\pm$ with the property that there exists a smooth, simple curve $\beta\colon (0,1) \to \mathbb{H}$ such that $\mathbb{H} \cap \partial A = \beta((0,1))$ we have that
$$\mathbb{P}[\gamma([0,\infty]) \cap A = \emptyset] = (\psi_A'(0))^{5/8}. \tag{0.1}$$
Show that (0.1) holds for all $A \in \mathcal{Q}_\pm$.
- Using the conformal invariance of Brownian motion, carefully justify (11.6) in the lecture notes.
- Carefully justify the last sentence in the proof of Theorem 11.3.