RANDOM PLANAR GEOMETRY, LENT 2020, EXAMPLE SHEET 3

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Problem 1. Suppose that $\gamma\colon [0,T] \to \overline{\mathbf{H}}$ is a simple curve (i.e., $s \neq t$ implies $\gamma(s) \neq \gamma(t)$) with $\gamma(0) = 0$ and $\gamma(t) \in \mathbf{H}$ for all $t \in (0,T]$. Show that $A_t = \gamma((0,t])$ for $t \in [0,T]$ is a family of locally growing compact $\mathbf{H}$-hulls. Show, moreover, that there exists a homeomorphism $\phi\colon [0,T] \to [0, \frac{1}{2}\mathrm{hcap}(A_T)]$ so that $\mathrm{hcap}(A_{\phi^{-1}(t)}) = 2t$ for all $t \in [0, \frac{1}{2}\mathrm{hcap}(A_T)]$. (This is the so-called capacity parameterization of γ.)

Problem 2. Suppose that $U\colon [0,T] \to \mathbf{R}$ is a continuous function. Let $g_t(z)$ solve the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Show for each $t \in [0,T]$ that g_t is a conformal transformation from its domain onto $\mathbf{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$ using the following steps.

- Show that $t \mapsto \mathrm{Im}(g_t(z))$ is decreasing in t, hence for each $z \in \mathbf{H}$, $t \mapsto g_t(z)$ is defined up until $\tau_z = \sup\{t \geq 0 : \mathrm{Im}(g_t(z)) > 0\}$. Conclude that $H_t = \{z : \tau_z > t\}$ is the domain of g_t.
- Show for each $t \in [0,T]$ that $z \mapsto g_t(z)$ is complex differentiable on H_t.
- Show for each $t \in [0,T]$ that $z \mapsto g_t(z)$ has an inverse defined on $\mathbf{H}$ by showing that $g_t(f_t(w)) = w$ for all $w \in \mathbf{H}$ where f_s for $s \in [0,t]$ solves the so-called *reverse chordal Loewner equation*

$$\partial_s f_s(w) = -\frac{2}{f_s(w) - U_{t-s}}, \quad f_0(w) = w.$$

Problem 3. Suppose that $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion and let (g_t) solve

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

- (Markov property) Suppose that τ is a stopping time for U which is almost surely finite and let $\widetilde{g}_t = g_{\tau+t}(g_\tau^{-1}(z + U_\tau)) - U_\tau$. Show that the maps $(\widetilde{g}_t)$ have the same distribution as the maps (g_t).
- (Scale invariance) Fix $r > 0$ and let $\widetilde{g}_t(z) = r g_{t/r^2}(z/r)$. Show that the maps $(\widetilde{g}_t)$ have the same distribution as the maps (g_t).

Suppose that D is a simply connected domain, $x, y \in \partial D$ are distinct, and $\varphi\colon \mathbf{H} \to D$ is a conformal transformation with $\varphi(0) = x$ and $\varphi(\infty) = y$. Explain why the definition of SLE_κ given by $\varphi(\gamma)$ where γ is an SLE_κ in $\mathbf{H}$ from 0 to ∞ is well-defined.

Problem 4.

- Suppose that B is a standard Brownian motion and $a < 0$. Show that $\sup_{t \geq 0}(B_t + at) < \infty$ almost surely.
- Suppose that (g_t) is the family of conformal maps which solve the Loewner equation with driving function $U_t = \sqrt{\kappa} B_t$ and, for each $x \in \mathbf{R}$, let $V_t^x = g_t(x) - U_t$ and $\tau_x = \inf\{t \geq 0 : V_t^x = 0\}$. For each $0 < x < y$, let $g(x,y) = \mathbf{P}[\tau_x = \tau_y]$. Show that if $g(1, 1+\epsilon/2) > 0$ for all $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$ then $g(x,y) > 0$ for all $0 < x < y$.

Problem 5. Fix $T > 0$ and let $D \subseteq \mathbf{H}$ be a simply connected domain. Suppose that $(A_t)_{t\in[0,T]}$ is a non-decreasing family of compact $\mathbf{H}$-hulls which are locally growing with $A_0 = \emptyset$, $\text{hcap}(A_t) = 2t$ for all $t \in [0,T]$, and $A_T \subseteq D$. Let $\psi\colon D \to \mathbf{H}$ be a conformal transformation which is bounded on bounded sets. Show that the family of compact $\mathbf{H}$-hulls $\widetilde{A}_t = \psi(A_t)$ for $t \in [0,T]$ is locally growing with $\widetilde{A}_0 = \emptyset$ and with

$$\text{hcap}(\widetilde{A}_t) = \int_0^t 2(\psi_s'(U_s))^2 ds \quad \text{where} \quad \psi_t = \widetilde{g}_t \circ \psi \circ g_t^{-1} \quad \text{for each} \quad t \in [0,T]$$

and $\widetilde{g}_t$ is the unique conformal transformation $\mathbf{H} \setminus \widetilde{A}_t \to \mathbf{H}$ with $\widetilde{g}_t(z) - z \to 0$ as $z \to \infty$.

Problem 6. In the setting of the previous problem, show that

$$\partial_t \psi_t(U_t) = \lim_{z \to U_t} \partial_t \psi_t(z) = -3\psi_t''(U_t).$$

Problem 7. Suppose that (A_t) is a non-decreasing family of $\mathbf{H}$-hulls which are locally growing and with $A_0 = \emptyset$. For each $t \geq 0$, let $a(t) = \text{hcap}(A_t)$ and assume that $a(t)$ is C^1. For each $t \geq 0$, let g_t be the unique conformal transformation which takes $\mathbf{H} \setminus A_t$ to $\mathbf{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$. Show that the conformal maps (g_t) satisfy the ODE:

$$\partial_t g_t(z) = \frac{\partial_t a(t)}{g_t(z) - U_t}, \quad g_0(z) = z$$

for some continuous, real-valued function U_t. (Hint: perform a time-change so that the hulls are parameterized by capacity, apply Loewner's theorem as proved in class, and then invert the time change.)

Problem 8. Suppose that B is a standard Brownian motion starting from $B_0 = x > 0$. For each $a \in \mathbf{R}$, let $\tau_a = \inf\{t \geq 0 : B_t = a\}$.

- For $a < x < b$, explain why $\mathbf{P}[\tau_a < \tau_b] = (b-x)/(b-a)$.
- Using the Girsanov theorem, explain why the law of B weighted by $B_{\tau_0 \wedge \tau_b}$ is equal to that of a BES^3 process stopped upon hitting b. That is, if $\mathbf{P}$ denotes the law of B and we define the law $\widetilde{\mathbf{P}}$ using the Radon-Nikodym derivative
$$\frac{d\widetilde{\mathbf{P}}}{d\mathbf{P}} = \frac{B_{\tau_0 \wedge \tau_b}}{\mathbf{E}[B_{\tau_0 \wedge \tau_b}]}$$
then the law of B under $\widetilde{\mathbf{P}}$ is that of a BES^3 process stopped upon hitting b.
- Explain why a standard Brownian motion conditioned to be non-negative is a BES^3 process.
- More generally, explain why a BES^d process with $d < 2$ conditioned to be non-negative is a BES^{4-d} process.

Problem 9. Suppose that (g_t) is the family of conformal maps associated with an SLE_κ with driving function U_t, i.e., $U_t = \sqrt{\kappa} B_t$ where B is a standard Brownian motion. Fix $z \in \mathbf{H}$ and let $z_t = x_t + iy_t = g_t(z)$. Assume that $\rho \in \mathbf{R}$ is fixed. Use Itô's formula to show that

$$M_t = |g_t'(z)|^{(8-2\kappa+\rho)\rho/(8\kappa)} y_t^{\rho^2/8\kappa} |U_t - z_t|^{\rho/\kappa}$$

is a continuous local martingale. (Hint: let

$$Z_t = \frac{(8-2\kappa+\rho)\rho}{8\kappa} \log g_t'(z) + \frac{\rho^2}{8\kappa} \log y_t + \frac{\rho}{\kappa} \log(U_t - z_t),$$

compute dZ_t using Itô's formula, take its real part, and exponentiate.)

Problem 10. Assume that we have the setup of Problem 9. Let $\Upsilon_t = y_t/|g_t'(z)|$. You may assume that

$$\frac{1}{4} \leq \frac{\Upsilon_t}{\mathrm{dist}(z, \gamma([0,t]) \cup \partial \mathbf{H})} \leq 4.$$

- Let $S_t = \sin(\arg(z_t - U_t))$. Explain why

$$M_t = |g_t'(z)|^{(8-\kappa+\rho)\rho/(4\kappa)} \Upsilon_t^{\rho(\rho+8)/(8\kappa)} S_t^{-\rho/\kappa}.$$

- By considering the above martingale with the special choice $\rho = \kappa - 8$, show that if $\kappa > 8$ then the SLE$_\kappa$ curve γ almost surely hits z. Conclude that γ fills all of $\mathbf{H}$. (Hint: recall that we showed in class that γ fills $\partial \mathbf{H}$. Deduce from this and the conformal Markov property that γ cannot separate z from ∞ without hitting it. Consider the behavior of S_t when γ is hitting a point on $\partial \mathbf{H}$ with either very large positive or negative values.)