On Hammersley's Method for One-Dimensional Covering Problems

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1. Personal History

I first met John Hammersley in Oxford during the years 1949–1952 when I held an ICI Fellowship at the Clarendon Laboratory. David Kendall used to run (with the aid of Pat Moran) a regular probability seminar in which he encouraged research workers in widely differing disciplines to participate. Hammersley then held an appointment in a department with the intriguing title *Lectureship in the Design and Analysis of Scientific Experiment*. David Finney was the lecturer from 1948–1955, and Michael Sampford and John Hammersley were his assistants. Their job was to provide mathematical, statistical, and computational advice to any of the science departments in Oxford that requested it. Hammersley had thus already begun his fruitful practice of "keeping open shop to all customers" and whenever he delivered a talk at the seminar, one could be sure of encountering a variety of stimulating new problems and ideas.

My wartime experience in radar research had introduced me to problems in geometrical probability, and whilst a graduate student at Cambridge I had published papers on the covering of a line by random intervals, and on the statistics of particle counters. It was useful for me to meet others who shared my interest, and a number of the problems discussed at the seminar were subsequently described in the monograph on *Geometrical Probability* by Maurice Kendall and Pat Moran (1963).

But my major research interest had moved to problems of lattice statistics, the Ising model, and order-disorder transitions in alloys. Moran (1947) had considered the statistical problem of the distribution of black-white joins in a lattice whose points could be black or white independently with probabilities p, $(1-p)$, and had proved that the distribution is normal. I felt it important to draw attention to the difference between the requirements of statistics and those of statistical mechanics. For the latter, the normality of the distribution gives *little* information of physical importance; the physicist, surprisingly, needs to determine all the higher moments and cumulants, and it is on the asymptotic form of these that the interesting critical behaviour depends.

For me personally one of the great benefits which I derived from the Oxford seminar was the introduction to the bright group of young statisticians who were active in organizing the research section of the Royal Statistical Society. Their public discussions and symposia were lively and challenging, and they cast their net widely. Two of the papers which Hammersley read to this section were *On Estimating Restricted Parameters* (1950) which dealt with problems for example in which the parameter sought was known to be an integer; and *Poor Man's Monte Carlo* (with K.W. Morton 1954) which discussed a Monte Carlo technique which did not require the use of large machines.

Much of the latter paper was devoted to a lattice model of a polymer molecule which took the excluded volume into account in a realistic way. In this paper Hammersley attributed the model of a random walk on a lattice which is not allowed to visit any site more than once to Meyer. He gave no reference and I was unable to trace to which Meyer he referred. In subsequent correspondence he suggested that it might have been J.E. Mayer the architect of the well known cluster integral theory of a condensing gas. I myself had been introduced to the model by G.S. Rushbrooke who presumably heard of it from his supervisor R.H. Fowler; in a letter to me Hammersley agreed that this may also have been his source.

Hammersley coined the term *self-avoiding walk* for the model, and this was adopted universally. Previously such walks had been described by a variety of names — *non-intersecting*, *non self-intersecting* and even *simple* (but of course they are far from simple). Curiously enough the terminology was challenged nearly thirty years later by Amit and his collaborators (1983). The original model envisaged selecting from the total ensemble of random walks these with no double or multiple points and giving equal weight to each of them. Amit et al. (1983) generated and analysed walks, which do not visit any site more than once, but whose probability of taking a step at any point is inversely proportional to the number of *unoccupied* neighbouring lattice sites; these they called *true self-avoiding walks.* The early Monte Carlo workers were careful *not* to general walks of this type (see e.g. Rosenbluth and Rosenbluth 1955) by weighting appropriately at the vertices. The argument is clearly one of semantics.

In a discussion remark following the above paper of Hammersley and Morton, Broadbent drew attention to a novel problem in which the randomness is associated with the *medium* rather than with the fluid. Subsequently he collaborated with Hammersley (1957) in providing a comprehensive formulation of this new class of problems which Hammersley described as *percolation processes*, a term which also gained universal acceptance. Some fifteen years later, when the important applications to solid state physics became apparent, the literature on percolation processes grew at an incredible rate.

My own approach to problems of lattice statistics had been to generate exact series expansions of substantial length for the logarithm of the partition function (the analogue of the cumulant generating function) and to use them to assess the asymptotic behaviour of the coefficients. This method had proved quite successful for the Ising model, for which a number of exact results were available by which the method could be checked. The same approach could be used to explore the behaviour of self-avoiding walks and percolation processes, and my research group at King's College established striking analogies between various features of these systems and thermodynamic properties of magnetic models.

I presented a paper to the Royal Statistical Society on these topics entitled *Some Statistical Problems Connected with Crystal Lattices* (1964) and was grateful for Hammersley's support in the discussion. Any talk of drawing conclusions from extrapolation arouses suspicion in the mind of the statistician. It was important to emphasize that our method was not just conjectured extrapolation; we made use of physical knowledge and insight to *postulate* an asymptotic form, and this postulate was tested and its parameters fitted by statistical data in a fairly standard manner.

By a simple argument involving sub-additive functions Hammersley had proved that the total number of self-avoiding walks of n steps on an infinite lattice was asymptotically of order μ^n and he called μ the *connective constant* (another term which gained wide acceptance). We were able to provide convincing statistical evidence that the total number of self-avoiding polygons of n steps was also asymptotically of order μ^n, and Hammersley subsequently established this result rigorously (1961).

For me one of the most amazing results of later research was the formulation in exact terms of the analogy we had discovered between self-avoiding walk models and percolation models and magnetic systems. In the n-vector model of ferromagnetism each site is occupied by an elementary magnetic spin which is free to rotate isotropically in n dimensions. $n = 1$ corresponds to the Ising model, $n = 2$ is called the x–y model, $n = 3$ the classical Heisenberg model. In 1972 de Gennes showed that $n = 0$ corresponds to the self-avoiding walk model.

One of my outstanding graduate students during my stay at Oxford was an Australian Rhodes Scholar named R.B. Potts. I had drawn his attention to a magnetic model with three orientations in a plane which had some properties analogous to those of the two orientation $\updownarrow$ Ising model. I thought the model might generalize to q-orientations in a plane. Potts demonstrated to me that the generalization which I sought was not as I had thought, q vectors in a plane, but in space, and the vectors must be such that the angle between any pair of them is the same. Potts published his results in a paper in the Proceedings of the Cambridge Philosophical Society (1952) since we considered the investigation to be an abstract mathematical exercise with little chance of physical application.

For nearly twenty years the Potts model was ignored. Then interest began to focus on magnetic models with different types of symmetry, and the number of papers on the Potts model grew with amazing rapidity. I can echo Hammersley's remarks (1983) "When children become adults, they embark on ideas and activities of which their parents are only dimly aware". Most surprising of all Kasteleyn and Fortuin (1969) demonstrated that the Potts model with $q = 1$ corresponds precisely to the percolation model.

In the present article I shall discuss problems arising from random intervals on a line in which Hammersley and I were interested in the late 1940's and early 1950's. I will relate these problems to one dimensional continuum percolation, a subject which has attracted interest and attention recently.

2. Statistics of Counters

The following problem arises when the finite resolving time of a recording apparatus is taken into account. Events are divided into two classes, recorded and unrecorded. Any recorded event is followed by a dead interval of length τ, during which any other event which occurs will be unrecorded. A typical example is an α-particle counter; a recorded particle causes the chamber to ionize, and no other particle can be recorded until the chamber has de-ionized. I dealt with this problem (Domb 1948) in the following manner.

Assume that the events are defined by a Poisson process, the probability of an event occurring in the interval $[y, y + dy]$ being λdy. Let $z_n(y)$ be the probability that n recorded events occur in $[0, y]$. It can be divided into mutually exclusive groups: (i) Those in which the recorder is live at point y, probability $z_{n1}(y)$. This means that no recorded event occurs in $[y - \tau, y]$, and hence n recorded events occur in $[0, y - \tau]$. (ii) Those in which the recorder is dead at point y, probability $z_{n2}(y)$. In this case a recorded event occurs in $[y - \tau, y]$.

It is now easy to construct equations for $z_n(y + dy)$ in terms of $z_{n1}(y)$ and $z_{n2}(y)$ leading to the following differential equation:

$$z'_n(y) = \lambda\big[z_{n-1}(y) - z_{n1}(y)\big]. \tag{2.1}$$

Thus, the function $z_{n1}(y)$ plays a key role in the structure of the equation.

When we look at $z_{n1}(y + dy)$ we see that all possibilities are covered by two cases:

(a) The recorder is live at y and remains live at $y + dy$; n recorded events occur in $[0, y]$, no event occurs in $[y, y + dy]$.

(b) The recorder is dead at y but becomes live at $y + dy$; $(n-1)$ recorded events occur in $[0, y-\tau]$, one event occurs in $[y-\tau, y-\tau+dy]$, and no event occurs in $[y, y+dy]$.

This gives rise to the differential equation:

$$z'_{n1}(y) = -\lambda z_{n1}(y) + \lambda z_{(n-1)1}(y-\tau). \tag{2.2}$$

I then showed that equations (2.1) and (2.2) are amenable to treatment by Laplace transforms, and that an explicit solution can readily be derived for $Z_n(p)$ the Laplace transform of $z_n(y)$. Moreover, the treatment can be generalized to a stochastic distribution of intervals $u(\tau)d\tau$. The only change required for this is the replacement of the second term on the right of (2.2) by the integral

$$\lambda \int_0^y u(\tau) z_{(n-1)1}(y-\tau)\, d\tau$$

and such a faltung can equally easily be handled by Laplace transforms.

A second type of instrument was used for recording events of a different kind which remains dead as long as events follow one another at intervals less than τ. This is closely related to the problem of covering a line by random intervals, which I had discussed previously (Domb 1947), again by means of Laplace transforms. I used my previous analysis to derive the distribution of recorded events for this second type of counter (Domb 1950), but noted that it was no longer a simple matter to generalize to a stochastic distribution. "The possibility of one interval completely covering another which follows it causes considerable mathematical complications."

A few years later Hammersley (1953) became interested in this second type of counter in connection with a device for counting blood cells electronically which had been developed in the Clinical Pathology Department of the Radcliffe Infirmary at Oxford. "A large number of blood cells, contained in a shallow chamber, are scanned by a photoelectric cell. The depth of the chamber and the concentration of blood cells in solution therein allow blood cells (supposed distributed at random through the chamber) to overlap when viewed from above the scanner. The field of view of the scanner at any instant is somewhat larger than the size of a blood cell, but is, nevertheless, of much the same order of magnitude. With passage of time the chamber moves underneath the photocell so that the field of view traces out a long narrow path not crossing or overlapping itself and only embracing a portion of the whole chamber. The blood cells have no motion relative to the chamber. As each blood cell comes under the photocell it produces an electrical impulse, whose duration depends upon the size and shape and orientation of the blood cell. These impulses go to a counter, which counts them except that it will not count any impulse which is overlapped by a previous impulse. The problem is to determine the number of blood cells in the chamber from a knowledge of the recorded count and the distribution of the lengths of individual impulses."

Hammersley came to discuss the problem with me, and I pointed out to him that there was no difficulty in calculating the mean, mean-square or any other moment; but I could not see how to provide a closed form solution. Hammersley worked on the problem and did in fact produce a complete solution. He made a handsome acknowledgement to me "I am very much indebted to Domb, who showed me how to surmount these difficulties by a brilliant application of the elementary theorem that the expectation of the sum of several (possibly independent) quantities is the sum of their expectations". In fact, he had achieved far more than I had ever thought possible.

A few years later Walter L. Smith (1957) re-derived Hammersley's results more neatly and concisely using the powerful methods of renewal theory. The Cambridge mathematician A.S. Besicovitch used to say "A mathematician's reputation rests on his bad proofs" (Burkill 1971). He wished to convey the idea that the originator of a result in mathematics usually establishes it by long and complicated proofs. This paves the way for the shorter and simpler proofs of later workers.

I wish to focus attention on one particular aspect of the solution, the probability that the portion $[0, y]$ of the line is completely covered. Hammersley incidentally provides a formal solution to this problem, but the expression he gives is complicated, and it seems to me that a direct attack on the problem itself, using his approach, yields a solution more readily.

3. Covering of a Line or Circle by Random Intervals[1]

When I returned to Cambridge in 1946 after radar-research for the Admiralty in World War 2, I brought with me the above covering problem with equal intervals. I needed to know whether anyone had tackled the problem previously, and Herman Bondi (who had been one of my colleagues at the Admiralty) referred me to Harold Jeffreys, whom he described as a mine of information on miscellaneous mathematical problems. Jeffreys immediately thought of the 'bicycle wheel problem' which he himself had formulated a few years previously as follows: A man is cycling along a road and passes through a region strewn with tacks; he wishes to know whether one has entered his tyre. Because of the traffic, he can only snatch glances at random times. At each glance he covers a fraction x of the wheel. What

[1] The remaining sections are adapted from a recent paper by the author in the Journal of Statistical Physics (Domb 1989). Detailed reference to other work on one-dimensional continuum percolation will be found in this paper. The author is indebted to Plenum Publishing Corporation for permission to reproduce some of this work.

FIG. 1. The bicycle wheel problem.

is the probability that after n glances he has covered the whole wheel? In mathematical terminology: n intervals are placed randomly on a circle, each covering a fraction x of the circle. What is the probability that the circle is completely covered (Figure 1)?

Jeffrey's drew my attention to a paper published by W.L. Stevens in 1939 in the Annals of Eugenics, entitled *Solution to a Geometrical Problem in Probability*, in which his problem was solved. Using a neat combinatorial argument, Stevens found for the probability $F(0)$ of complete coverage

$$F(0) = 1 - \binom{n}{1}(1-x)^{n-1} + \binom{n}{2}(1-2x)^{n-1} - \binom{n}{3}(1-3x)^{n-1} + \cdots \quad (3.1)$$

the series terminating at the kth term, k being the integral part of $1/x$. Stevens also derived a formula for $F(i)$, the probability that there are i gaps on the circle.

In 1929, R.A. Fisher published an article entitled *Tests of Significance in Harmonic Analysis*, in which he calculated the probability that the largest interval in the random division of a circle is less than x (Figure 2). When Stevens's solution for $F(0)$ appeared, Fisher noted that it was identical with his, and a moment's reflection is enough to convince one that the two problems are identical. Fisher pointed this out in a note published in 1940.

But surprisingly, R.A. Fisher, one of the founders of the modern theory of statistics, was unaware that the distribution of length of the largest interval in the random division of a line had been correctly solved by Whitworth many years before, and was reproduced in his classic book, *Choice and Chance* (solutions to problems 666 and 667 published in 1897).

Problem 666: A line of length c is divided into n segments by $n-1$ random points. Find the chance that no segment is less than a given length a, where $c > na$ (say, $c - na = ma$).

Problem 667: In the last question find the chance that r of the segments shall be less than a and $n - r$ greater than a.

FIG. 2. Random division of a circle.

More precise dating of the solutions will be discussed in the next section.

4. Whitworth's Choice and Chance

We will preface this section with a few biographical details relating to Whitworth, taken from the Dictionary of National Biography (1901–1911, p. 655) and will continue with some comments on the different editions of his famous publication *Choice and Chance*.

William Allen Whitworth was born in 1840, and entered St. John's College as a undergraduate in October 1858. His performance in the Mathematics Tripos was not distinguished — he was 16th Wrangler in 1862 — but this does not seem to have represented his true ability. While still an undergraduate he was principal editor of the *Oxford, Cambridge and Dublin Messenger of Mathematics*, started at Cambridge in November 1861. The publication was continued as *The Messenger of Mathematics*; Whitworth remained one of the editors till 1880, and was a frequent contributor.

After leaving Cambridge in 1862 he was successively chief mathematics master at Portarlington School and Rossal School, and professor of mathematics at Queen's College, Liverpool (1862, 1864); he was a fellow of St. John's College from 1867 to 1882. At the same time Whitworth followed a second career of distinction in the Church, being ordained deacon in 1865 and priest in 1866. He held appointments as a curate at three churches in Liverpool from 1865 to 1875, and as vicar of two churches in London from 1875 until his death in 1905.

The first edition of *Choice and Chance* was published in 1867 while he was in Liverpool, and was a reproduction of lectures given to ladies in Queen's College Liverpool in the Michaelmas term of 1866. The book was subtitled *Two Chapters in Arithmetic*, and its aims, as described in the Preface, were modest enough:

I had already discovered that the usual method of treating ques-

> tions of selection and arrangement was capable of modification and so great simplification, that the subject might be placed on a purely arithmetical basis; and I deemed that nothing would better serve to furnish the exercise which I desired for my classes, and to elicit and encourage a habit of exact reasoning, than to set before them, and establish as an application of arithmetic, the principles on which such questions of "choice and chance" might be solved.

He expressed the hope that his publication might be of service "in conducing to a more thoughtful study of arithmetic than is common at present; extending the perception and recognition of the important truth, that arithmetic, or the art of counting, demands no more science than good and exact common sense".

Chapter 1 was devoted to "Choice", and was followed by 24 questions; Chapter 2 to "Chance", followed by 20 questions. The questions were all arithmetical in character. An appendix was devoted to *Permutations and Combinations Treated Algebraically*: "In my experience as a teacher I have found the proofs here set forth more intelligible to younger students than those given in the text books in common use". Whitworth here derived a number of standard elementary combinatorial formulae, and ended with a new combinatorial proof of the binomial theorem.

The second edition, published only three years later (1870) from St. John's College, Cambridge, added three appendices containing more sophisticated material. Appendix II was devoted to *Distributions* (into different groups or parcels), Appendix III to *Derangements*: "a series of propositions are given which are not usually found in text books of algebra. But I can see no reason why examples of such simple propositions ... should be excluded from elementary treatises in which more complex but essentially less important theorems find place". Appendix IV was concerned with the celebrated St. Petersburg problem and its background. More than 100 miscellaneous new examples were added.

In the third edition, published in 1878, the material in the appendices was revised and enlarged, and incorporated into the main text. There were now four chapters on *Choice* and four chapters on *Chance*, the final, brief eighth chapter carrying the title, *The Geometrical Representation of Chances*, the number of examples was increased to 300, and they were divided into different classes. The Preface contained the proclamation, "Questions requiring the application of the Integral Calculus are not included in the book, which only fulfills its title to be an Elementary Treatise".

In the fourth edition, published in 1886, the number of examples grew to 640, and a new chapter in the *Choice* section was added dealing with problems where the order in which gains and losses occur is relevant, e.g., if there is a condition that losses must never exceed gains. A short additional chapter in the *Chance* section entitled, *The Rule of Succession*, was devoted to a precise treatment of situations in which the probability of an event is supposed completely unknown, but the results of a number of trials are available. What can now be predicted about future trials?

The fifth and final edition was not published until 1901. But in 1897 there appeared a volume entitled, *DCC Exercises in Choice and Chance*. which provided fairly detailed solutions to the 640 examples of the fourth edition, and to 60 new examples, several of which were concerned with the random division of a line by a number of points. Questions 667 and 668, which were quoted in the previous section, are included among the latter. The preface to the fifth edition, which now contained 1000 examples, described the new category as follows: "A new feature will be recognized in a class of problems which found scarcely any place in former editions; the class which includes investigations into the mean value of the largest part, (or the smallest, or any other in order of magnitude) or of functions of such a part, when a magnitude is divided at random".

It is clear that Whitworth was actively working on this type of problem at the time. Quoting again from the same preface, "the most important addition in the body of the work is the very far-reaching theorem ... which enables us to write down at sight the mean value of such functions as α^3, $\alpha^3\beta^4$, $\alpha\beta\gamma$ etc. when $\alpha, \beta, \gamma, \ldots$ are the parts into which a given magnitude is divided at random. I first published this theorem in a pamphlet in the year 1898". The calculations of quantities of this type given in the *DCC Exercises* volume did not make use of the theorem, and were much longer.

From the above discussion it is clear that the problem with which we are concerned was tackled by Whitworth at some date between 1886 and 1897, most probably close to the latter date.

5. Whitworth's Solution

Whitworth divided the line into a number of discrete segments, which would eventually be allowed to become very large. He then used standard combinatorial formulae which he had developed in the text to enumerate various cases outlined in examples 666 and 667 (see Section 3).

We shall retain Whitworth's notation for historical reasons, but shall find it convenient to use generating functions to reproduce his combinatorial formulae. Whitworth assumed that the line of length c was divided into ωc equal elements. The given length a would then contain ωa elements. Take a dummy variable x_1 to enumerate the possible configurations of the first segment, x_2 the second segment, $\ldots$, x_n the nth segment. Then the generating function which enumerates all configurations in any division of the line by $n-1$ points is

$$F(t, x_1, x_2, \ldots, x_n) = (tx_1+t^2x_1^2+\cdots)(tx_2+t^2x_2^2+\cdots)\cdots(tx_n+t^2x_n^2+\cdots) \tag{5.1}$$

assuming no two points are identical. The total number of segments is ωc and therefore all possible configurations are enumerated by the coefficient of $t^{\omega c}$ in $F(t; x_1, \ldots, x_n)$. If we need the total number of configurations, we put $x_1 = x_2 = \cdots = x_n = 1$ and find the coefficient $t^{\omega c-n}$ in $(1-t)^{-n}$, which is

$$\binom{\omega c-n+n-1}{n-1} = \binom{\omega c-1}{n-1} = \frac{(\omega c-1)(\omega c-2)\cdots(\omega c-n+1)}{(n-1)!}. \tag{5.2}$$

For problem 666 one needs to enumerate all configurations with each of the segments containing ωa or more elements, and Whitworth realized that this was identical with finding all possible configurations which divide a line of length $c - n\omega a$ into n parts. This is clear from the generating function approach, since the appropriate enumerator is now

$$\begin{aligned} &t^{\omega a}x_1^{\omega a}(1+tx_1+t^2x_1^2+\cdots)t^{\omega a}x_2^{\omega a}(1+tx_2+t^2x_2^2+\cdots)\cdots \\ &\qquad\qquad \times t^{\omega a}x_n^{\omega a}(1+tx_n+t^2x_n^2+\cdots). \end{aligned} \tag{5.3}$$

We therefore require the coefficient of $t^{\omega(c-na)}$, i.e., of $t^{\omega ma}$ ($ma = c - na$) in $(1-t)^{-n}$, which is

$$\binom{\omega ma+n-1}{n-1} = \frac{(\omega ma+n-1)(\omega ma+n-2)\cdots(\omega ma+1)}{(n-1)!}. \tag{5.4}$$

Hence the probability that no segment is less than a is found by taking the quotient of (5.4) by (5.2) and is equal to

$$\frac{(\omega ma+n-1)(\omega ma+n-2)\cdots(\omega ma+1)}{(\omega c-1)(\omega c-2)\cdots(\omega c-n+1)}. \tag{5.5}$$

When ω increases indefinitely, this reduces to

$$[ma/c]^{n-1} \tag{5.5$'$}$$

For example 667, Whitworth pointed out that all orders of choice of the r segments less than a, and the $n-r$ segments greater than a, give rise to the same number of configurations, and we can therefore deal with the case in which the r segments are at the beginning and the $n-r$ at the end, and multiply by $\binom{n}{r}$. The enumerating generating function is then

$$\begin{aligned} &(tx_1+t^2x_1^2+\cdots+t^{\omega a-1}x_1^{\omega a-1})(tx_2+t^2x_2^2+\cdots+t^{\omega a-1}x_2^{\omega a-1})\cdots \\ &\times (tx_r+t^2x_r^2+\cdots+t^{\omega a-1}x_r^{\omega a-1})t^{\omega a}x_{r+1}^{\omega a}(1+tx_{r+1}+t^2x_{r+1}^2+\cdots) \\ &\times t^{\omega a}x_{r+2}^{\omega a}(1+tx_{r+2}+t^2x_{r+2}^2+\cdots)\cdots t^{\omega a}x_n^{\omega a}(1+tx_n+t^2x_n^2+\cdots). \end{aligned} \tag{5.6}$$

FIG. 3. Random intervals on a line.

The total number of configurations is the coefficient of $t^{\omega ma+\omega ra-r}$ in

$$\frac{(1-t^{\omega a-1})^r}{(1-t)^r}(1-t)^{-(n-r)} = (1-t^{\omega a-1})^r(1-t)^{-n}. \tag{5.7}$$

Expanding the first factor by the binomial theorem, we derive the series

$$\begin{aligned}&\binom{n+\omega(m+r)a-r-1}{n-1} - \binom{r}{1}\binom{n+\omega(m+r-1)a-r}{n-1}\\ &\quad+\binom{r}{2}\binom{n+\omega(m+r-2)a-r+1}{n-1} - \cdots\\ &\quad+(-1)^s\binom{r}{s}\binom{n+\omega(m+r-s)a-r+s-1}{n-1} + \cdots.\end{aligned} \tag{5.8}$$

In the limit of very large ω this simplifies very considerably; dividing by (5.2) and taking the limit, we obtain

$$\begin{aligned}&\left(\frac{m+r}{m+n}\right)^{n-1} - \binom{r}{1}\left(\frac{m+r-1}{m+n}\right)^{n-1} + \binom{r}{2}\left(\frac{m+r-2}{m+n}\right)^{n-1} + \cdots\\ &\quad+(-1)^s\binom{r}{s}\left(\frac{m+r-s}{m+n}\right)^{n-1} + \cdots + (-1)^r\left(\frac{m}{m+n}\right)^{n-1}.\end{aligned} \tag{5.9}$$

Expression (5.9) must be multiplied by $\binom{n}{r}$ to obtain the complete solution.

Although (5.8) looks complicated, the generating function (5.7) from which it is derived is quite simple, and the calculation of averages and higher moments can be undertaken by standard routine.

The probability of complete coverage, with which we have been concerned, corresponds to $r = n$, and is given by

$$\begin{aligned}&1 - \binom{n}{1}\left(\frac{c-a}{c}\right)^{n-1} + \binom{n}{2}\left(\frac{c-2a}{c}\right)^{n-1} + \cdots\\ &\qquad\qquad+(-1)^s\binom{n}{s}\left(\frac{c-sa}{c}\right)^{n-1} + \cdots\end{aligned} \tag{5.10}$$

the series terminating at the last term before $c - sa$ becomes negative.

The solutions given above are the same as those derived later by Fisher (1929) and Stevens (1939), with the slight adaptation needed for a problem on a circle rather than on a line.

6. Use of a Poisson Process: Equal Intervals

The problem to be considered is the following (Figure 3).

Events occur at random on a line in a Poisson distribution, the probability of an occurrence in $[y, y+dy]$ being λdy. Each event is the left-hand end of an interval of length τ. Choose any section $[0, y]$ of the line. Calculate the probability $z(y)$ that the section is completely covered.

We divide $z(y)$ into mutually exclusive classes $z(y, \xi)$ in which the last event occurred between $y - \xi$ and $y - \xi - d\xi$. Then if $y > \tau$, ξ cannot be greater than τ or the section $[0, y]$ would not be covered. Also, $z(y, \xi)$ can be decomposed into three independent contributions: (i) No event occurs in $[y-\xi, y]$; (ii) an event occurs in $[y-\xi-d\xi, y-\xi]$; (iii) the section $[0, y-\xi]$ is covered. Hence, we deduce that

$$z(y) = \int_0^\tau z(y, \xi)\, d\xi = \int_0^\tau \lambda e^{-\lambda\xi} z(y-\xi)\, d\xi \quad (y > \tau). \tag{6.1}$$

If $y \leq \tau$, we must take into account the additional possibility that an event occurs in $[y-\tau, 0]$, and no event occurs in $[0, y]$; we easily find that

$$z(y) = \int_0^y \lambda e^{-\lambda\xi} z(y-\xi)\, d\xi + e^{-\lambda y} - e^{-\lambda\tau} \quad (y \leq \tau). \tag{6.2}$$

Taking Laplace transforms in y in (6.1) and (6.2), we derive for the Laplace transform $Z(p)$ of $z(y)$,

$$Z(p) = \frac{p(1-e^{-\lambda\tau}) - \lambda e^{-\lambda\tau}(1-e^{-p\tau})}{p + \lambda e^{-(p+\lambda)\tau}}. \tag{6.3}$$

If the denominator is expanded as $[1 + (\lambda/p)e^{-(p+\lambda)\tau}]^{-1}$ and the terms are interpreted individually, the combinatorial solution is obtained. If further the solution is broken down into mutually exclusive classes in which exactly n events occur in $[0, y]$, the identity

$$z(y) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} f_n(y) \tag{6.4}$$

can be deduced, where $f_n(y)$ is the probability for n events. In this way the solution of Whitworth, Fisher, and Stevens can be simply derived.

FIG. 4. Solution of $xe^{-x} = \beta e^{-\beta}$ giving asymptotic decay.

But if we are interested in large y/τ, the asymptotic behaviour of $z(y)$ is determined by the zeros of the denominator of (6.3), i.e. by solutions $-\gamma$ of

$$q + \beta e^{-(\beta+q)} = 0 \quad (q = p\tau, \beta = \lambda\tau). \tag{6.5}$$

There is only one real root, $-\gamma$, which dominates the asymptotic behaviour, the complex roots providing transients which rapidly decay. γ is the solution other than β of the equation

$$xe^{-x} = \beta e^{-\beta} \tag{6.6}$$

(see Figure 4). We then find the asymptotic solution

$$z(y) \sim \frac{e^{-\beta}(\beta - \gamma)}{\gamma(1-\gamma)} e^{-\gamma\nu} \quad (y = \nu\tau). \tag{6.7}$$

When β is large (high density of events), γ is small, and when β is small, γ is large. The probability of an infinite cluster of overlapping intervals in a one-dimensional percolating system is zero; equation (6.7) describes the approach to zero as a finite system grows large.

The calculation for $z_k(y)$, the probability that the line contains k clusters, follows similar lines. The integral equation is now

$$z_k(y) = \begin{cases} \int_0^y \lambda e^{-\lambda\xi} z_k(y-\xi)\, d\xi & (\xi \le \tau) \\ \int_0^y \lambda e^{-\lambda\xi} z_{k-1}(y-\xi)\, d\xi & (\xi > \tau) \end{cases} \tag{6.8}$$

with special treatment for $k = 1$. Taking Laplace transforms, we find

$$Z_k(p) = \frac{\lambda e^{-\tau(p+\lambda)}}{p + \lambda e^{-\tau(p+\lambda)}} Z_{k-1}(p) = \left(\frac{\lambda e^{-\tau(p+\lambda)}}{p + \lambda e^{-\tau(p+\lambda)}} \right)^{k-1} Z_1(p). \tag{6.9}$$

FIG. 5. Stochastic distribution: covering of intervals.

From this it can be deduced that the asymptotic distribution of clusters, in the limit of large ν $(= y/\tau)$, is normal with mean $\nu\beta e^{-\beta}$ and variance $\nu[\beta e^{-\beta} - 2\beta^2 e^{-2\beta}]$.

The calculation of $W(x, y)dx$, the probability that the covered portion of the line is between x and $x+dx$, is more complicated, and the distribution contains δ-function terms corresponding to various discrete probabilities. The moments of the distribution can be calculated in a straightforward manner. For example,

$$\begin{aligned}\langle x\rangle &= y(1-e^{-\beta})\\ \langle x^2\rangle &= y^2(1-e^{-\beta}) - e^{-\beta}\left(y^2 - \frac{2y}{\lambda} + \frac{2}{\lambda^2}\right)\\ &\quad + e^{-2\beta}\left[(y-\tau)^2 - \frac{2(y-\tau)}{\lambda} + \frac{2}{\lambda^2}\right].\end{aligned} \tag{6.10}$$

7. Stochastic Distribution of Intervals

When the intervals are not all equal the previous method breaks down because an early event can overlap a later one (Figure 5). The behaviour at the point y is no longer dependent only on the latest event at $y - \xi_1$, but all previous events at $y - \xi_1, y - \xi_1 - \xi_2, \ldots,$ must be considered. The way in which to deal with this new situation was demonstrated by Hammersley, as we mentioned above in Section 2.

Assume a probability distribution of intervals $u(\tau)d\tau$, and divide $z(y)$ into mutually exclusive classes as follows:

$$z(y) = z(y; \xi_1) + z(y; \xi_1, \xi_2) + z(y; \xi_1, \xi_2, \xi_3) + \cdots + z_0(y) \tag{7.1}$$

where $z(y; \xi_1)$ represents the class in which the point y is covered by the last event at $y - \xi_1$, $z(y; \xi_1, \xi_2)$ represents the class in which the point is *not covered* by the *last event* at $y - \xi_1$, but is covered by the *last but one* at $y - \xi_1 - \xi_2$; $z(y; \xi_1, \xi_2, \xi_3)$ represents the class in which the point y is *not covered* by the *last two events*, but is covered by the *last but two* at $y - \xi_1 - \xi_2 - \xi_3$; $z_0(y)$ represents the class in which *no covering event* occurs in $[0, y]$ but the point y is covered by an event occurring *before*. Write

$$U(\tau) = \int_0^\tau u(t)\, dt \tag{7.2}$$

which represents the probability of an interval of length not exceeding τ; $1 - U(\tau)$ then represents the probability of an interval greater than τ. It is easy to derive the following relations (Figure 5):

$$\begin{aligned}
z(y; \xi_1) &= \int_0^y \lambda e^{-\lambda \xi}[1 - U(\xi_1)] z(y - \xi_1)\, d\xi_1 \quad (0 < \xi_1 < y) \\
z(y; \xi_1, \xi_2) &= \iint \lambda e^{-\lambda \xi_1} U(\xi_1)\, d\xi_1\, \lambda e^{-\lambda \xi_2}[1 - U(\xi_1 + \xi_2)] \\
&\quad \times d\xi_2\, z(y - \xi_1 - \xi_2) \quad (0 < \xi_1, \xi_2 < y, \xi_1 + \xi_2 < y) \\
z(y; \xi_1, \xi_2, \xi_3) &= \iiint \lambda e^{-\lambda \xi_1} U(\xi_1)\, d\xi_1\, \lambda e^{-\lambda \xi_2} U(\xi_2)\, d\xi_2\, \lambda e^{-\lambda \xi_3} \\
&\quad \times [1 - U(\xi_1 + \xi_2 + \xi_3)]\, d\xi_3\, z(y - \xi_1 - \xi_2 - \xi_3) \\
&\qquad (0 < \xi_1, \xi_2, \xi_3 < y, \xi_1 + \xi_2 + \xi_3 < y).
\end{aligned} \tag{7.3}$$

To see the structure of these relations, it is convenient to transform to new variables,

$$\eta_1 = \xi_1,\ \eta_2 = \xi_1 + \xi_2,\ \eta_3 = \xi_1 + \xi_2 + \xi_3, \ldots \tag{7.4}$$

so that the limits of integration in the new variables are

$$0 < \eta_1 < \eta_2 < \eta_3 < \cdots < y. \tag{7.5}$$

We then find

$$\begin{aligned}
z(y; \eta_1) &= \int_0^y \lambda e^{-\lambda \eta_1}[1 - U(\eta_1)] z(y - \eta_1)\, d\eta_1 \\
z(y; \eta_1, \eta_2) &= \iint \lambda^2 U(\eta_1)\, d\eta_1\, e^{-\lambda \eta_2}[1 - U(\eta_2)] z(y - \eta_2)\, d\eta_2 \\
z(y; \eta_1, \eta_2, \eta_3) &= \iiint \lambda^3 U(\eta_1) d\eta_1 U(\eta_2) d\eta_2\, e^{-\lambda \eta_2}[1 - U(\eta_3)] z(y - \eta_3) d\eta_3.
\end{aligned} \tag{7.6}$$

The integration in η_1 in $z(y; \eta_1, \eta_2)$ yields a function of η_2. Similarly, the integrations of η_1, η_2 in $z(y; \eta_1, \eta_2, \eta_3)$ yield a function of η_3. The structure of equation (7.1) is therefore

$$z(y) = \int_0^y v(\eta) z(y - \eta)\, d\eta + z_0(y) \tag{7.7}$$

which is still of the form amenable to Laplace transforms. The function $v(\eta)$ can be calculated by summing the successive contributions in (7.6).

However, we shall use a shortcut to evaluating $v(\eta)$ by considering a related problem, the probability $\zeta(y)$ that the point y is *covered* by an event occurring in $[0, y]$. We can decompose $\zeta(y)$ in a similar manner to (7.1)–(7.6) and we obtain the same integrals without the $z(y-\eta)$ factors, i.e.,

$$\zeta(y) = \int_0^y v(\eta)\, d\eta. \tag{7.8}$$

But the probability $1 - \zeta(y)$ that the point y is *not covered* by an event occurring in $[0, y]$ was calculated in an elementary manner by Hammersley (1953) to be

$$\exp\left[-\lambda y + \lambda \int_0^y U(t)\, dt\right]. \tag{7.9}$$

The derivation is straightforward. Let us call an event which occurs in $[0, y]$ and covers the point y a *covering event*. The probability that a covering event does not occur in the interval $[y - \xi - d\xi, y - \xi]$ is

$$\exp\bigl\{-\lambda[1 - U(\xi)]\, d\xi\bigr\}. \tag{7.10}$$

But all such intervals from $\xi = 0$ to $\xi = y$ are independent. Hence the probability that *no* covering event occurs in $[0, y]$ is the product of factors of type (7.10) from $\xi = 0$ to $\xi = y$, and this leads directly to (7.9). Hence we can derive $v(y)$ by differentiating (7.8),

$$v(y) = \lambda e^{-\lambda y}[1 - U(y)] \exp\left[\lambda \int_0^y U(t)\, dt\right]. \tag{7.11}$$

On examining (7.11) and comparing with (7.5) and (7.6), it is not difficult to see how the formula could be derived directly, the successive terms in (7.6) corresponding to successive terms in the expansion of $\exp[\lambda \int_0^y U(t)dt]$.

It is convenient to introduce a function $\overline{U}(y)$ which is the complement of $U(y)$,

$$U(y) + \overline{U}(y) = 1. \tag{7.12}$$

Relations (7.9) and (7.11) assume a simplified form in terms of $\overline{U}(y)$ as follows:

$$1 - \zeta(y) = \exp\left[-\lambda \int_0^y \overline{U}(t)\, dt\right], \tag{7.13}$$

$$v(y) = \lambda \overline{U}(y) \exp\left[\lambda \int_0^y \overline{U}(t)\, dt\right]. \tag{7.14}$$

For a distribution $u(\tau)d\tau$ which is zero for $\tau \geq \tau_0$, $\overline{U}(y)$ is also zero for $\tau \geq \tau_0$; for a long-range distribution, $\overline{U}(y)$ provides a direct representation of the tail.

The solution of (7.7) by Laplace transforms is very simple in principle, and gives for the Laplace transform $Z(p)$ of $z(y)$

$$Z(p) = \frac{Z_0(p)}{1 - V(p)} \tag{7.15}$$

where $V(p)$ is the Laplace transform of $v(y)$. As in Section 6 the asymptotic behaviour of $z(y)$ is determined by the roots of the denominator of (7.15), and we shall find close parallels to the behaviour for equal intervals.

8. Distributions with a Finite Mean Value

It is important to discuss the general behaviour of the function $V(p)$ as p decreases from $+\infty$ through zero to $-\infty$. First note that $v(y)$ is positive for all y. Hence

$$V(p) = \int_0^\infty v(y)e^{-py}\,dy \tag{8.1}$$

increases monotonically as p decreases. Thus, there can be only one real root of the equation $V(p) = 1$.

We illustrate this behaviour by reconsidering the case of equal intervals, for which

$$u(t) = \delta(t - \tau) \tag{8.2}$$

$$v(y) = \begin{cases} \lambda e^{-\lambda y} & y \leq \tau \\ 0 & y > \tau \end{cases} \tag{8.3}$$

$$V(p) = \frac{\lambda}{p + \lambda}\left[1 - e^{-\tau(p+\lambda)}\right]. \tag{8.4}$$

For large positive p, $V(p)$ is small; as p decreases to zero, $V(p)$ rises to $(1 - e^{-\lambda\tau})$; and at $p = 0$, it is therefore less than 1; for negative p, it continues its steady increase, becoming 1 at a unique negative value $-\gamma/\tau$; it then increases exponentially for large p.

Let us now consider a general distribution with a finite cutoff τ_0. From (7.14) we see that $v(y)$ is zero for $y > \tau_0$. The general pattern of behaviour is similar to that for equal intervals, the value for $p = 0$ being given, from (8.1), by

$$V(0) = \int_0^\infty v(y)\,dy. \tag{8.5}$$

Using (7.8) and (7.13), we find that

$$V(0) = 1 - \exp\left[-\lambda \int_0^\infty \overline{U}(t)\,dt\right]. \tag{8.6}$$

But

$$\int_0^\infty \overline{U}(t)\,dt = \left[t\overline{U}(t)\right]_0^\infty + \int_0^\infty tu(t)\,dt = \overline{\tau} \tag{8.7}$$

which is the average length of interval. Therefore

$$V(0) = 1 - e^{-\lambda\overline{\tau}} \tag{8.8}$$

which is again less than 1. Hence $V(p)$ reaches the value 1 for a negative value of $p = -\overline{\gamma}/\overline{\tau}$, and by analogy with (6.7) the asymptotic behaviour of $z(y)$ is an asymptotic decay, $\exp(-\overline{\gamma}y/\overline{\tau})$. The probability of the line $[0, y]$ being covered tends to zero for large y, i.e., there is no percolating cluster.

Now consider a distribution with a long tail of the form

$$u(\tau) \sim \frac{A}{\tau^s}. \tag{8.9}$$

Then

$$\overline{U}(y) = \int_y^\infty u(\tau)\,d\tau \sim \frac{A}{(s-1)y}. \tag{8.10}$$

Reverting to equation (8.7), the integral on the left-hand side exists if $s > 2$, $\overline{\tau}$ is defined, and the equation remains valid. Hence the argument of the previous paragraph can be repeated, and there is no percolating cluster.

The argument can be extended to a distribution of the form

$$u(\tau) \sim \frac{A}{\tau^2(\ln\tau)^s} \quad (s > 1) \tag{8.11}$$

for which the integral of $tu(t)$ converges to give a finite mean value $\overline{\tau}$. We now have

$$\overline{U}(y) \sim \frac{A}{(s-1)y(\ln y)^{s-1}} \tag{8.12}$$

and equation (8.7) is still valid. Again there is no percolating cluster for large y. The argument applies equally for

$$u(\tau) \sim \frac{A}{\tau^2(\ln\tau)(\ln\ln\tau)^s}, \frac{A}{\tau^2\ln\tau(\ln\ln\tau)(\ln\ln\ln\tau)^s}, \ldots \quad (s > 1) \tag{8.13}$$

the general conclusion being that as long as the mean interval of the distribution is finite, $z(y)$ decays exponentially for large y.

9. Distributions with an Infinite Mean Value

For a distribution $u(\tau)$ for which the integral of $tu(t)$ does not converge, i.e., for which $\overline{\tau}$ becomes infinite, the argument of the previous section would indicate that $V(0)$, which is equal to $1 - e^{-\lambda\overline{\tau}}$, becomes equal to 1. Hence, from (7.15) the dominating term in the asymptotic behaviour of $z(y)$ will no longer be an exponential decay, but a constant. Therefore the system should now have a percolating cluster.

We can use the argument of Section 7 to specify in more detail what happens. Consider the probability that the point y is not covered by an event which has occurred in $[-y_0, 0]$. Using equation (7.10), we see that this probability is given by

$$\exp\left[-\lambda \int_{y}^{y+y_0} \overline{U}(\xi)\, d\xi\right]. \tag{9.1}$$

But for any of the distributions of the previous section for which $\overline{\tau}$ is infinite [(8.9) with $s \leq 2$; (8.11) and (8.13) with $s \leq 1$] the integral of $\overline{U}(\xi)$ diverges, and by choosing y_0 sufficiently large, (9.1) can be made as small as we please. Hence there is probability 1 that the point y is covered by an event occurring before 0, i.e., that the interval $[0, y]$ is completely covered by such an event. This corresponds to a percolating cluster.

We therefore find that with such distributions percolation occurs however small the value of λ, so that the system becomes critical however small the percolation probability.

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