On a Problem of Straus

P. Erdős and A. Sárközy[1]

1. Introduction and Results

Throughout this paper we use the following notation: $\{x\}$ denotes the fractional part of x. The distance from x to the nearest integer is denoted by $\|x\|$: $\|x\| = \min(\{x\}, 1 - \{x\})$. The cardinality of the finite set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. The counting functions of the finite sets $\mathcal{A}, \mathcal{B}, \ldots$ of non-negative integers are denoted by $A(x), B(x), \ldots$ so that, e.g., $A(x) = |\mathcal{A} \cap \{1, 2, \ldots, [x]\}|$. If $\mathcal{A}$ is a finite or infinite set of integers, then let $\mathcal{P}(\mathcal{A})$ denote the set of distinct integers n that can be represented in the form $n = \sum_{a \in \mathcal{A}} \epsilon_a a$ where $\epsilon_a = 0$ or 1 for all a and $0 < \sum_{a \in \mathcal{A}} \epsilon_a < \infty$. If $\mathcal{A}$ is a set of integers such that no a_i is the average of any subset of $\mathcal{A}$ consisting of two or more elements, then $\mathcal{A}$ is said to be non-averaging.

Erdős and Straus raised the question of deciding the maximum cardinality $f(N)$ of a non-averaging subset of $\{0, 1, 2, \ldots, N\}$? This problem has been studied by Abbott (1976, 1980, 1986), Bosznay (1989), Straus (1968), and Erdős and Straus (1970), and the best estimates are due to Bosznay (1989) and Erdős and Straus (1970) who proved that

$$f(N) \gg N^{1/4} \tag{1.1}$$

and

$$f(N) \ll N^{2/3}, \tag{1.2}$$

respectively. Furthermore, Straus (1968) reduced the upper estimate of $f(N)$ to the following problem: what is the maximum number $k = F(N)$ such that there exist two subsets $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}, \mathcal{B} = \{b_1, b_2, \ldots, b_k\}$ of $\{0, 1, 2, \ldots, N\}$ so that the sums of non-empty subsets of $\mathcal{A}$ are different from the sums of non-empty subsets of $\mathcal{B}$, i.e., $\mathcal{P}(\mathcal{A}) \cap \mathcal{P}(\mathcal{B}) = \emptyset$? He conjectured that the maximum number $F(N)$ is attained when $\mathcal{A}, \mathcal{B}$ are of the form $\mathcal{A} = \{0, 1, \ldots, k-1\}$ and $\mathcal{B} = \{N-k+1, N-k+2, \ldots, N\}$ for an optimal k. This construction leads to

$$F(N) \geq \left[(2N)^{1/2}\right] - 1. \tag{1.3}$$

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Furthermore, he proved that

$$f(N) \le 2F(N) + 1. \tag{1.4}$$

In this paper, our goal is to give an upper bound for $F(N)$ and thus, by (1.4), also for $f(N)$. In fact, we will prove

Theorem 1. *For $N > N_0$ we have*

$$F(N) < 201(N \log N)^{1/2}. \tag{1.5}$$

This is only by a factor $(\log N)^{1/2}$ worse than the conjectured $F(N) \ll N^{1/2}$ and probably also this $(\log N)^{1/2}$ factor could be eliminated by improving on a lemma (Lemma 1) in our proof; we will return to this problem.

Combining Theorem 1 with (1.4) we obtain:

Corollary 1. *For $N > N_0$ we have*

$$f(N) < 403(N \log N)^{1/2}.$$

(Compare with (1.2).)

In the second half of this paper our goal is to study the infinite analogue of this problem: if $\mathcal{A}, \mathcal{B}$ are infinite sets of positive integers such that

$$\mathcal{P}(\mathcal{A}) \cap \mathcal{P}(\mathcal{B}) = \emptyset, \tag{1.6}$$

then how large can $\min(A(x), B(x))$ be? Of course, Theorem 1 implies that $\min(A(x), B(x)) \ll x^{1/2}$. We conjecture that (1.6) implies

$$\liminf_{x\to\infty} \frac{\min(A(x), B(x))}{x^{1/2}} = 0 \tag{1.7}$$

and, perhaps, even

$$\liminf_{x\to\infty} \frac{A(x)B(x)}{x} = 0$$

holds; unfortunately, we have not been able to prove this. On the other hand, we will prove that $x^{1/2}$ in the denominator in (1.7) cannot be replaced by $x^{1/2}(\log x)^{-1/2-\epsilon}$:

Theorem 2. *Let $\beta_1, \beta_2, \dots$ be an infinite sequence of positive real numbers with*

$$\sum_{n=1}^{\infty} \beta_n < \frac{1}{2}. \tag{1.8}$$

Then there exist two infinite sets $\mathcal{A} = \{a_1, a_2, \dots\}, \mathcal{B} = \{b_1, b_2, \dots\}$ *of distinct positive integers such that*

$$\max(a_n, b_n) \le 8n\beta_n^{-1} \quad \text{for } n = 1, 2, \dots \tag{1.9}$$

and

$$\mathcal{P}(\mathcal{A}) \cap \mathcal{P}(\mathcal{B}) = \emptyset. \tag{1.10}$$

Thus, e.g., choosing $\beta_n = c\big(n \log n (\log\log n)^{1+\epsilon}\big)^{-1}$ here (where $\epsilon > 0$ and c is a positive constant small enough in terms of ϵ) we obtain that there exist infinite sets $\mathcal{A}, \mathcal{B}$ of positive integers such that

$$\liminf_{x\to\infty} \frac{\min(A(x), B(x))}{x^{1/2}(\log x)^{-1/2}(\log\log x)^{-1/2-\epsilon}} > 0$$

and (1.10) holds.

On the other hand, it is easy to see that there exist infinite sets $\mathcal{A}, \mathcal{B}$ of positive integers such that (1.10) holds and both $\mathcal{A}$ and $\mathcal{B}$ have positive upper density. In fact, to see this let $x_n = 2^{2^n}$, and let

$$\mathcal{A} = \bigcup_{k=1}^{\infty} \{x_{2k}, x_{2k}+1, \dots, [3x_{2k}/2]\}$$

and

$$\mathcal{B} = \bigcup_{k=1}^{\infty} \{x_{2k+1}, x_{2k+1}+1, \dots, [3x_{2k+1}/2]\}.$$

It can be shown easily that these sets $\mathcal{A}, \mathcal{B}$ have the desired properties.

Also, it would be interesting to decide how fast $A(x)B(x)$ can grow for infinite sets $\mathcal{A}, B$ satisfying (1.10). In the construction above we have

$$A(x)B(x) \gg x^{3/2}$$

for infinitely many x. Perhaps, this inequality is nearly best possible.

Let $\mathcal{A} = \{a_1, a_2, \dots\}, \mathcal{B} = \{b_1, b_2, \dots\}$ be infinite increasing sequences of positive real numbers with the property that

$$\left| \sum_{i=1}^{\infty} \epsilon_i a_i - \sum_{i=1}^{\infty} \epsilon_i' b_i \right| \ge 1$$

whenever $\epsilon_i = 0$ or 1 for all i, $\epsilon_i' = 0$ or 1 for all i, $0 < \sum_{i=1}^{\infty} \epsilon_i < \infty$ and $0 < \sum_{i=1}^{\infty} \epsilon_i' < \infty$. Perhaps, these assumptions imply that

$$\liminf_{x\to\infty} \frac{\max(A(x), B(x))}{x} = 0$$

and

$$\lim_{x\to\infty} \frac{\min(A(x), B(x))}{x} = 0$$

(where $A(x) = |\{i : a_i \le x\}|, B(x) = |\{i : b_i \le x\}|$); we have not been able to prove this.

2. Two Lemmas

The proof of Theorem 1 will be based on the following result of Sárközy (1989):

LEMMA 1. *Let N be a positive integer with $N > 2500$, let $\mathcal{A} \subset \{1, 2, \dots, N\}$ and*

$$|\mathcal{A}| > 100(N \log N)^{1/2}.$$

Then there are integers d, y, z such that

$$1 \le d < 10^4 N|\mathcal{A}|^{-1},$$

$$z > 7^{-1}10^{-4}|\mathcal{A}|^2,$$

$$y < 7 \cdot 10^4 Nz|\mathcal{A}|^{-2}$$

and

$$\{yd, (y+1)d, \dots, zd\} \subset \mathcal{P}(\mathcal{A}).$$

We need one more lemma:

LEMMA 2. *Let M, N, t, d be positive integers with $M \le N$,*

$$d \le t \le N, \tag{2.1}$$

and let

$$\mathcal{A} \subset \{M, M+1, \dots, N\}, \tag{2.2}$$

$$|\mathcal{A}| = t. \tag{2.3}$$

Then for every integer u with

$$0 \le u \le M(t-d), \tag{2.4}$$

there is an integer s such that

$$u \le s < u + Nd, \tag{2.5}$$

$$d \mid s \tag{2.6}$$

and s can be written in the form

$$s = \sum_{a \in \mathcal{A}} \epsilon_a a \quad \textit{where} \quad \epsilon_a = 0 \quad \textit{or} \quad 1 \quad \textit{for all} \quad a \tag{2.7}$$

(so that either $s = 0$ or $s \in \mathcal{P}(\mathcal{A})$).

Proof of Lemma 2: It suffices to show that there are integers $x_0, x_1, \ldots, x_r$ such that $x_0 = 0$,

$$x_{i-1} < x_i \leq x_{i-1} + Nd \quad \text{for} \quad i = 1, 2, \ldots, r, \tag{2.8}$$

$$x_r > M(t-d),$$

$$d \mid x_i \quad \text{for} \quad i = 0, 1, \ldots, r \tag{2.9}$$

and

$$x_i \in \mathcal{P}(\mathcal{A}) \quad \text{for} \quad i = 1, 2, \ldots, r. \tag{2.10}$$

In fact, if $x_0, x_1, \ldots, x_r$ are defined in this way and u satisfies (2.4), then there is an x_i with $u \leq x_i < u + Nd$ so that (2.5), (2.6), and (2.7) hold with x_i in place of s.

These numbers $x_0, x_1, \ldots, x_r$ can be defined recursively. Let $x_0 = 0$. Assume that $x_0, x_1, \ldots, x_i$ $(i \geq 0)$ have been defined with the desired properties and

$$x_i \leq M(t-d). \tag{2.11}$$

Then by (2.10) (and $x_0 = 0$) there is a subset $\mathcal{A}_1 \subset \mathcal{A}$ $(\mathcal{A}_1 = \emptyset$ for $i = 0)$ such that

$$\sum_{a \in \mathcal{A}_1} a = x_i. \tag{2.12}$$

By (2.1), this implies

$$x_i \geq \sum_{a \in \mathcal{A}_1} M = |\mathcal{A}_1| M. \tag{2.13}$$

It follows from (2.11) and (2.13) that

$$|\mathcal{A}_1| \leq t - d. \tag{2.14}$$

Let us write $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$ so that, by (2.3) and (2.14),

$$|\mathcal{A}_2| = |\mathcal{A}| - |\mathcal{A}_1| = t - |\mathcal{A}_1| \geq d.$$

Let $\mathcal{A}_3$ be a subset of $\mathcal{A}_2$ with $|\mathcal{A}_3| = d$. Then there is a non-empty subset $\mathcal{A}_4$ of $\mathcal{A}_3$ with

$$d \mid \sum_{a \in \mathcal{A}_4} a. \tag{2.15}$$

(In fact, if $\mathcal{A}_3 = \{a_1, a_2, \dots, a_d\}$, then either there is a k with $a_1 + a_2 + \cdots + a_k \equiv 0 \pmod{d}$ so that we may choose $\mathcal{A}_4 = \{a_1, a_2, \dots, a_k\}$, or there are k, l with $k < l$, $a_1 + a_2 + \cdots + a_l \equiv a_1 + a_2 + \cdots + a_k \pmod{d}$ so that $\mathcal{A}_4 = \{a_{k+1}, a_{k+2}, \dots, a_l\}$ can be chosen.) Let

$$x_{i+1} = x_i + \sum_{a \in \mathcal{A}_4} a. \tag{2.16}$$

Then by (2.9) and (2.15) we have $d \mid x_{i+1}$. Furthermore, $x_{i+1} \in \mathcal{P}(\mathcal{A})$ follows from (2.10) and (2.16). Finally, by (2.2) and (2.16) we have

$$\begin{aligned} x_i < x_{i+1} &\le x_i + \sum_{a \in \mathcal{A}_4} N \\ &= x_i + N|\mathcal{A}_4| \le x_i + N|\mathcal{A}_3| \\ &= x_i + Nd \end{aligned}$$

and this completes the proof of Lemma 2. ■

3. Completion of the Proof of Theorem 1

We have to show that $N > N_0$,

$$\begin{aligned} \mathcal{A} &= \{a_1, a_2, \dots, a_k\} \subset \{1, 2, \dots, N\}, \\ \mathcal{B} &= \{b_1, b_2, \dots, b_k\} \subset \{1, 2, \dots, N\} \end{aligned} \tag{3.1}$$

(where $a_1 < a_2 < \cdots < a_k,\ b_1 < b_2 < \cdots < b_k$) and

$$k \ge 201(N \log N)^{1/2} \tag{3.2}$$

imply that

$$\mathcal{P}(\mathcal{A}) \cap \mathcal{P}(\mathcal{B}) \ne \emptyset. \tag{3.3}$$

We may assume that $a_{[k/2]+1} \le b_{[k/2]+1}$. Let us write

$$\begin{aligned} M &= a_{[k/2]+1}, \\ \mathcal{A}' &= \{a_2, a_3, \dots, a_{[k/2]+1}\}, \\ \mathcal{B}' &= \{b_{[k/2]+1}, b_{[k/2]+2}, \dots, b_k\} \end{aligned}$$

so that, in view of (3.1) and (3.2) for $N > N_0$ we have

$$N > M = a_{[k/2]+1} \ge k/2 > 100(N \log N)^{1/2}, \tag{3.4}$$

$$\mathcal{A}' \subset \{1, 2, \dots, M\}, \tag{3.5}$$

$$|\mathcal{A}'| = [k/2] > 100(N \log N)^{1/2} \ge 100(M \log M)^{1/2}, \tag{3.6}$$

$$\mathcal{B}' \subset \{M, M+1, \dots, N\} \tag{3.7}$$

and

$$|\mathcal{B}'| = k - [k/2] \geq \frac{k}{2}. \tag{3.8}$$

By (3.4), (3.5) and (3.6), for large N we may apply Lemma 1 with M and $\mathcal{A}'$ in place of N and $\mathcal{A}$, respectively. We obtain that there exist integers d, y, z such that

$$1 \leq d < 10^4 M|\mathcal{A}'|^{-1}, \tag{3.9}$$
$$z > 7^{-1}10^{-4}|\mathcal{A}'|^2, \tag{3.10}$$
$$y < 7 \cdot 10^4 Mz|\mathcal{A}'|^{-2} \tag{3.11}$$

and

$$\{yd, (y+1)d, \ldots, zd\} \subset \mathcal{P}(\mathcal{A}'). \tag{3.12}$$

To prove (3.3), it suffices to show that

$$\mathcal{P}(\mathcal{A}') \cap \mathcal{P}(\mathcal{B}') \neq \emptyset. \tag{3.13}$$

If there is a positive integer s such that

$$yd \leq s \leq zd, \tag{3.14}$$
$$d \mid s \tag{3.15}$$

and

$$s \in \mathcal{P}(B'), \tag{3.16}$$

then by (3.12), also $s \in \mathcal{P}(\mathcal{A}')$ holds so that $s \in \mathcal{P}(\mathcal{A}') \cap \mathcal{P}(\mathcal{B}')$ whence (3.13) follows. Thus it suffices to show that there is a positive integer s satisfying (3.14), (3.15) and (3.16). To prove this, we are going to apply Lemma 2 with $\mathcal{B}'$, $|\mathcal{B}'| = k - [k/2]$ and yd in place of $\mathcal{A}$, t and u, respectively. Then (2.2) holds by (3.7). Furthermore, by (3.1), (3.2), (3.6), (3.8) and (3.9) we have

$$t = |\mathcal{B}'| \geq \frac{k}{2} \tag{3.17}$$

and

$$\begin{aligned} d &< 10^4 M|\mathcal{A}'|^{-1} \leq 10^4 N\big(100(N \log N)^{1/2}\big)^{-1} \\ &= 100N^{1/2}(\log N)^{-1/2} = o(k) \end{aligned} \tag{3.18}$$

so that also (2.1) holds. Finally, it follows from (3.5), (3.6), (3.11) and (3.12) that

$$\begin{aligned} u = yd &< 7 \cdot 10^4 Mz|\mathcal{A}'|^{-2} d \\ &= 7 \cdot 10^4 M|\mathcal{A}'|^{-2}(zd) < 7 \cdot 10^4 M|\mathcal{A}'|^{-2}|\mathcal{A}'|M \\ &= 7 \cdot 10^4 M^2|\mathcal{A}'|^{-1} < 7 \cdot 10^4 M^2\big(100(M \log M)^{1/2}\big)^{-1} \\ &= 700M^{3/2}(\log M)^{-1/2} \end{aligned} \tag{3.19}$$

and by (3.2), (3.17) and (3.18), for large N we have

$$\begin{aligned} t-d &\geq \frac{k}{2}-o(k) > \frac{k}{3} \\ &> 60(N\log N)^{1/2} \\ &\geq 60(M\log M)^{1/2}. \end{aligned} \tag{3.20}$$

(2.4) follows from (3.19) and (3.20). Thus, in fact, all the assumptions in Lemma 2 hold so that the lemma can be applied. We obtain that there is an integer s such that

$$u = yd \leq s < yd + Nd, \tag{3.21}$$

$$d \mid s \tag{3.22}$$

and

$$s \in \mathcal{P}(\mathcal{B}'). \tag{3.23}$$

(Note that $s \neq 0$ by $s \geq yd > 0$.)

It follows from (3.6), (3.10), (3.11) and (3.21) that

$$\begin{aligned} s &< (y+N)d < \big(7\cdot 10^4 Mz|\mathcal{A}'|^{-2}+N\big)d \\ &< \big\{(7\cdot 10^4 Nz\big(100(N\log N)^{1/2}\big)^{-2} + 10^{-4}(\log N)^{-1}|\mathcal{A}'|^2\big\}d \\ &= \big(o(z)+o(z)\big)d = o(zd). \end{aligned} \tag{3.24}$$

(3.15) and (3.16) hold by (3.22) and (3.23) while (3.14) follows from (3.21) and (3.24), and this completes the proof of Theorem 1. ■

4. Proof of Theorem 2

We are going to define the sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ recursively.

Let $\alpha = (\sqrt{5}+1)/2$. Let a_1 and b_1 be the least positive integers a and b such that $0 < \{a\alpha\} < \beta_1$ and $1-\beta_1 < \{b\alpha\} < 1$, respectively. If $a_1, a_2. \dots, a_{n-1}$ and $b_1, b_2, \dots, b_{n-1}$ have been defined, then let a_n and b_n be the least positive integers a and b such that

$$0 < \{a\alpha\} < \beta_n, \quad a \notin \{a_1, a_2, \dots, a_{n-1}\}$$

and

$$1-\beta_n < \{b\alpha\} < 1, \quad b \notin \{b_1, b_2, \dots, b_{n-1}\},$$

respectively.

First we are going to prove (1.10). If $\mathcal{A}'$ is a finite (non-empty) subset of $\mathcal{A}$, then in view of (1.8) we have

$$0 < \sum_{a_i \in \mathcal{A}'} \{a_i \alpha\} < \sum_{a_i \in \mathcal{A}'} \beta_i < \sum_{i=1}^{\infty} \beta_i < \frac{1}{2}$$

whence

$$0 < \sum_{a_i \in \mathcal{A}'} \{a_i \alpha\} = \left\{ \Big(\sum_{a_i \in \mathcal{A}'} a_i \Big) \alpha \right\} < \frac{1}{2}. \tag{4.1}$$

Furthermore, it follows from (1.8) and the definition of the set $\mathcal{B}$ that if $\mathcal{B}'$ is a finite (non-empty) subset of $\mathcal{B}$, then we have

$$\begin{aligned} |\mathcal{B}'| = \sum_{b_i \in \mathcal{B}'} 1 &> \sum_{b_i \in \mathcal{B}'} \{b_i \alpha\} > \sum_{b_i \in \mathcal{B}'} (1 - \beta_i) \\ &= |\mathcal{B}'| - \sum_{b_i \in \mathcal{B}'} \beta_i > |\mathcal{B}'| - \sum_{i=1}^{\infty} \beta_i > |\mathcal{B}'| - \frac{1}{2} \end{aligned}$$

whence

$$\frac{1}{2} < \left\{ \sum_{b_i \in \mathcal{B}'} \{b_i \alpha\} \right\} = \left\{ \Big(\sum_{b_i \in \mathcal{B}'} b_i \Big) \alpha \right\} < 1. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\sum_{a_i \in \mathcal{A}'} a_i \neq \sum_{b_i \in \mathcal{B}'} b_i$$

which proves (1.10).

To prove (1.9), we need the following lemma:

Lemma 3. *Let $\alpha = (\sqrt{5}+1)/2$. If δ is a real number with $0 < \delta < 1$ and x, y are arbitrary real numbers, then there is an integer m such that*

$$x < m \leq x + 4\delta^{-1} \tag{4.3}$$

and

$$\|m\alpha - y\| < \delta. \tag{4.4}$$

Proof: This can be proved by using standard tools of the theory of continued fractions (see, e.g., Hardy and Wright 1960); for the sake of completeness we give the proof. Let $q_o = 1, q_1 = 1, \ldots, q_n = q_{n-1} + q_{n-2}, \ldots$ denote the Fibonacci numbers. These numbers are the denominators of the convergents of the continued fraction expansion of α so that for all n there is an integer p_n such that

$$\left|\alpha - \frac{p_n}{q_n}\right| < q_n^{-2} \quad (\text{for } n = 0, 1, 2, \dots). \tag{4.5}$$

Clearly, $q_n = q_{n-1} + q_{n-2} \leq 2q_{n-1}$ for $n \geq 2$. Thus there is an integer k with

$$\frac{2}{\delta} < q_k \leq \frac{4}{\delta}. \tag{4.6}$$

Then we have

$$x < [x] + i \leq x + 4\delta^{-1} \quad \text{for} \quad i = 1, 2, \dots, q_k. \tag{4.7}$$

Write $j = \big[q_k(y - [x]\alpha)\big]$ so that

$$\left|\frac{j}{q_k} - \big(y - [x]\alpha\big)\right| < \frac{1}{q_k}. \tag{4.8}$$

Define the integer i_j by

$$i_j p_k \equiv j \pmod{q_k}, \;\; 1 \leq i_j \leq q_k \tag{4.9}$$

and write $m = [x] + i_j$. Then (4.3) holds by (4.7), and it follows from (4.5), (4.6), (4.8) and (4.9) that

$$\begin{aligned} \|m\alpha - y\| &= \big\|\big([x] + i_j\big)\alpha - y\big\| \\ &= \left\|\frac{i_j p_k}{q_k} + i_j\left(\alpha - \frac{p_k}{q_k}\right) + \big([x]\alpha - y\big)\right\| \\ &= \left\|\frac{j}{q_k} - \big(y - [x]\alpha\big) + i_j\left(\alpha - \frac{p_k}{q_k}\right)\right\| \\ &\leq \left|\frac{j}{q_k} - \big(y - [x]\alpha\big)\right| + |i_j|\left|\alpha - \frac{p_k}{q_k}\right| \\ &< \frac{1}{q_k} + q_k \cdot \frac{1}{q_k^2} \\ &= \frac{2}{q_k} < \delta \end{aligned}$$

so that also (4.4) holds and this completes the proof of the lemma. ■

Now we are going to prove (1.9). By the construction of the sets $\mathcal{A}, \mathcal{B}$, it suffices to show that there are at least n integers a and at least n integers b such that

$$\big|\{a : 0 < a \leq 8n\beta_n^{-1}, \; 0 < \{a\alpha\} < \beta_n\}\big| \geq n \tag{4.10}$$

and

$$\left|\{b : 0 < b \le 8n\beta_n^{-1},\ 1 - \beta_n < \{b\alpha\} < 1\}\right| \ge n, \tag{4.11}$$

respectively.

To prove (4.10), it suffices to show that for $i = 0, 1, \ldots, n-1$, there is an integer a such that

$$8i\beta_n^{-1} < a \le 8(i+1)\beta_n^{-1},\ 0 < \{a\alpha\} < \beta_n. \tag{4.12}$$

In fact, applying Lemma 3 with $\beta_n/2, 8i\beta_n^{-1}$ and $\beta_n/2$ in place of δ, x and y, respectively, we obtain that there is an integer m such that

$$x = 8i\beta_n^{-1} < m \le x + 4\delta^{-1} = 8i\beta_n^{-1} + 4(\beta_n/2)^{-1} = 8(i+1)\beta_n^{-1}$$

and

$$\left\| m\alpha - \frac{\beta_n}{2} \right\| < \frac{\beta_n}{2}$$

whence

$$\{m\alpha\} < \beta_n$$

(and $\{m\alpha\} > 0$ since α is irrational) which proves (4.12). Similarly, applying Lemma 3 with $\beta_n/2, 8i\beta_n^{-1}$ and $1 - \beta_n/2$ in place of δ, x and y, respectively, we obtain that there is an integer b with

$$8i\beta_n^{-1} < b \le 8(i+1)\beta_n^{-1},\ 1 - \beta_n < \{b\alpha\} < 1$$

which implies (4.11) and this completes the proof of the theorem. ∎

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Mathematical Institute
Hungarian Academy of Sciences
Reáltanoda ul. 13–15
Budapest.