**Theorem 2.3 of** *The Random-Cluster Model* **corrected**

**This replaces the correction dated 30 Sep 2006**

I am grateful to Kenshi Hosaka for pointing out a serious problem with Theorem 2.3. This theorem is used at three places later in the book (RCM), and an alternative argument suffices at each such place. The details follow.

Inequalities (2.4)–(2.5) do not in general imply (2.2), and Hosaka's counterexample is as follows. Take $S = \{1, 2, 3\}$ and $\Omega = \{0, 1\}^S$. Let

$$\mu_2(100) = \mu_2(010) = \mu_2(001) = \tfrac{1}{8},$$
$$\mu_2(110) = \mu_2(101) = \mu_2(011) = \tfrac{7}{48},$$
$$\mu_2(000) = \tfrac{1}{16}, \quad \mu_2(111) = \tfrac{1}{8},$$

and define $\mu_1$ symmetrically by $\mu_1(xyz) = \mu_2([1-x][1-y][1-z])$. It may be checked that $\mu_1$, $\mu_2$ satisfy (2.4)–(2.5), whereas

$$\mu_2(111)\mu_1(000) < \mu_1(001)\mu_2(110).$$

The alleged 'proof' of Theorem 2.3 contains a mathematical error (as well as a spelling error) near the middle of page 24 of RCM, where it is claimed that it suffices to consider the case $b \geq 2$. One must in fact consider the other case as well, and this is where the problem lies.

Here is a corrected version of Theorem 2.3.

**Theorem 2.3″.** *Let $\mu_1$, $\mu_2$ be a pair of strictly positive probability measures on $(\Omega, \mathcal{F})$ such that*

$$\text{(2.4)} \qquad \mu_2(\omega^e)\mu_1(\omega_e) \geq \mu_1(\omega^e)\mu_2(\omega_e), \qquad \omega \in \Omega,\ e \in E.$$

*If, in addition, either $\mu_1$ or $\mu_2$ satisfies*

$$\text{(2.5}'\text{)} \qquad \mu(\omega^{ef})\mu(\omega_{ef}) \geq \mu(\omega^e_f)\mu(\omega^f_e), \qquad \omega \in \Omega,\ e, f \in E,$$

*then (2.2) holds.*

*Proof.* Let $\mu$ be a strictly positive probability measure satisfying (2.5′). We show first that $\mu$ satisfies (2.2) with $\mu_1 = \mu_2 = \mu$, that is

$$\text{(2.6}'\text{)} \qquad \mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1)\mu(\omega_2).$$

We shall prove this by induction on the Hamming distance $H(\omega_1, \omega_2)$. Inequality (2.6′) is a triviality when: either $H(\omega_1, \omega_2) = 1$, or the $\omega_i$ are ordered (in that either $\omega_1 \leq \omega_2$, or vice versa). The only non-trivial case with $H(\omega_1, \omega_2) = 2$ is of the form: $\omega_1 = \omega_f^e$, $\omega_2 = \omega_e^f$ where $e$, $f$ are distinct edges. This is handled by assumption (2.5′).

Let $h \geq 3$ and suppose that (2.6′) holds for all pairs $\omega_1, \omega_2$ satisfying $H(\omega_1, \omega_2) < h$. Let $\omega_1, \omega_2 \in \Omega$ be such that $H(\omega_1, \omega_2) = h$, and furthermore such that neither $\omega_1 \leq \omega_2$ nor $\omega_1 \geq \omega_2$. There exist integers $a, b$ such that $a, b \geq 1$ and $a + b = h$, and disjoint subsets $A, B \subseteq E$ with cardinalities $a$ and $b$ respectively, such that:

$$\begin{aligned}
\text{if } e \in A, &\quad (\omega_1(e), \omega_2(e)) = (1, 0),\\
\text{if } e \in B, &\quad (\omega_1(e), \omega_2(e)) = (0, 1),\\
\text{if } e \in E \setminus (A \cup B), &\quad \omega_1(e) = \omega_2(e).
\end{aligned}$$

We fix an ordering $(e_i : i = 1, 2, \ldots, |E|)$ of the set $E$ in which edges in $A$ are indexed $1, 2, \ldots, a$, and edges in $B$ are indexed $a + 1, a + 2, \ldots, a + b$. A configuration $\omega$ may be written as a 'word' $\omega(e_1) \cdot \omega(e_2) \cdot \ldots \cdot \omega(e_{|E|})$; we write $0^x$ for a sub-word of length $x$ every entry of which is 0, with a similar meaning for $1^y$. Since the entries of the configurations $\omega_1, \omega_2, \omega_1 \vee \omega_2, \omega_1 \wedge \omega_2$ are constant off $A \cup B$, we shall omit explicit reference to these values. Thus, for example, $\omega_1 = 1^a \cdot 0^b$ and $\omega_2 = 0^a \cdot 1^b$.

Since $h = a + b \geq 3$, either $a \geq 2$ or $b \geq 2$, and it suffices by symmetry to assume $a \geq 2$. By the induction hypothesis,

$$\begin{aligned}
\mu(1^{a+b})\mu(0^{a-1} \cdot 1 \cdot 0^b) &\geq \mu(1^a \cdot 0^b)\mu(0^{a-1} \cdot 1^{b+1})\\
&\qquad \text{since } H(1^a \cdot 0^b, 0^{a-1} \cdot 1^{b+1}) = h - 1,\\
\mu(0^{a-1} \cdot 1^{b+1})\mu(0^{a+b}) &\geq \mu(0^{a-1} \cdot 1 \cdot 0^b)\mu(0^a \cdot 1^b)\\
&\qquad \text{since } H(0^{a-1} \cdot 1 \cdot 0^b, 0^a \cdot 1^b) = b + 1 < h,
\end{aligned}$$

whence

$$\begin{aligned}
\mu(1^{a+b})\mu(0^{a-1} \cdot 1 \cdot 0^b)\mu(0^{a+b}) &\geq \mu(1^a \cdot 0^b)\mu(0^{a-1} \cdot 1^{b+1})\mu(0^{a+b})\\
&\geq \mu(1^a \cdot 0^b)\mu(0^{a-1} \cdot 1 \cdot 0^b)\mu(0^a \cdot 1^b).
\end{aligned}$$

Therefore,

$$\mu(1^{a+b})\mu(0^{a+b}) \geq \mu(1^a \cdot 0^b)\mu(0^a \cdot 1^b),$$

and the induction step is complete.

We now use a telescoping argument. We identify a configuration $\omega \in \Omega$ with the set of indices $\eta(\omega)$ at which $\omega$ takes the value 1. Let $\xi_1, \xi_2 \in \Omega$, and write $A_k = \eta(\xi_k)$. Let $B = A_1 \setminus A_2 = \{b_1, b_2, \ldots, b_r\}$, and write $B_s = \{b_1, b_2, \ldots, b_s\}$ for $s \geq 1$. Assume $\xi_1 \neq \xi_2$, and without loss of generality that $r \geq 1$. By (2.4),

$$\begin{aligned}
\frac{\mu_2(\xi_1 \vee \xi_2)}{\mu_2(\xi_2)} &= \frac{\mu_2(A_2 \cup B_r)}{\mu_2(A_2 \cup B_{r-1})} \cdot \frac{\mu_2(A_2 \cup B_{r-1})}{\mu_2(A_2 \cup B_{r-2})} \cdots \frac{\mu_2(A_2 \cup B_1)}{\mu_2(A_2)}\\
&\geq \frac{\mu_1(A_2 \cup B_r)}{\mu_1(A_2 \cup B_{r-1})} \cdot \frac{\mu_1(A_2 \cup B_{r-1})}{\mu_1(A_2 \cup B_{r-2})} \cdots \frac{\mu_1(A_2 \cup B_1)}{\mu_1(A_2)}\\
&= \frac{\mu_1(\xi_1 \vee \xi_2)}{\mu_1(\xi_2)}.
\end{aligned}$$

If $\mu_1$ satisfies (2.5′), then it satisfies (2.6′), and (2.2) follows with $\xi_i = \omega_i$.

Similarly, if $\mu_2$ satisfies (2.5′), then it satisfies (2.6′), and (2.2) follows with $\xi_1 \vee \xi_2 = \omega_1, \xi_2 = \omega_1 \wedge \omega_2$. □

*Proof of Theorem 2.6.* Theorem 2.6 is correct as stated, but the given proof refers to Theorem 2.3 and therefore requires a small patch. As shown in RCM, (2.7) is equivalent to

$$(2.14') \qquad \frac{\mu_2(\zeta^e)}{\mu_2(\zeta_e)} \geq \frac{\mu_1(\xi^e)}{\mu_1(\xi_e)}, \qquad \xi \leq \zeta.$$

It is elementary that (2.2) implies (2.14′).

Suppose conversely that (2.14′) holds, and use the telescoping argument at the end of the last proof. In the notation used there, by (2.14′),

$$\begin{aligned}
\frac{\mu_2(\xi_1 \vee \xi_2)}{\mu_2(\xi_2)} &= \frac{\mu_2(A_2 \cup B_r)}{\mu_2(A_2 \cup B_{r-1})} \cdot \frac{\mu_2(A_2 \cup B_{r-1})}{\mu_2(A_2 \cup B_{r-2})} \cdots \frac{\mu_2(A_2 \cup B_1)}{\mu_2(A_2)} \\
&\geq \frac{\mu_1((A_1 \cap A_2) \cup B_r)}{\mu_1((A_1 \cap A_2) \cup B_{r-1})} \cdot \frac{\mu_1((A_1 \cap A_2) \cup B_{r-1})}{\mu_1((A_1 \cap A_2) \cup B_{r-2})} \\
&\qquad\qquad \cdots \frac{\mu_1((A_1 \cap A_2) \cup B_1)}{\mu_1(A_1 \cap A_2)} \\
&= \frac{\mu_1(\xi_1)}{\mu_1(\xi_1 \wedge \xi_2)}
\end{aligned}$$

as required. □

*Proof of Theorem 2.19.* This follows as in RCM, with the reference to Theorem 2.3 replaced by reference to Theorem 2.3″ above. □

*Proof of Theorem 3.79.* In order to apply Theorem 2.3″ on page 60 of RCM, we must check that the law of either $R$ or $S$ satisfies (2.5′). Both claims are in fact true. It is standard (and straightforward) to check this for $S$.

As for $R$, we show below that

$$(1) \qquad \mu_1(A^{xy})\mu_1(A) \geq \mu_1(A^x)\mu_1(A^y), \qquad A \subseteq V,\ x, y \in V \setminus A,\ x \neq y.$$

Let $A \subseteq V$, $x, y \in V \setminus A$, $x \neq y$. Let $a$ (respectively, $c$) be the number of edges of the form $\langle x, z\rangle$ (respectively, $\langle y, z\rangle$) with $z \in A$, let $b$ (respectively, $d$) be the number of edges of the form $\langle x, z\rangle$ (respectively, $\langle y, z\rangle$) with $z \notin A$ and $z \neq x, y$, and let $e$ be the number of edges joining $x$ and $y$. By (3.76),

$$\frac{\mu_1(A^x)}{\mu_1(A)} = \frac{(1-p)^{b+e} Z_{\overline{A^x}}(p, q-1)}{(1-p)^a Z_{\overline{A}}(p, q-1)} = \frac{\phi(x \text{ is isolated})}{(1-p)^a (q-1)},$$

where $\phi = \phi_{\overline{A}, p, q-1}$. Similarly,

$$\frac{\mu_1(A^{xy})}{\mu_1(A)} = \frac{\phi(x, y \text{ are isolated})}{(1-p)^{a+c+e}(q-1)^2}.$$

Therefore, the ratio of the left to the right sides of (1) is

$$(1-p)^{-e}\frac{\phi(x, y \text{ are isolated})}{\phi(x \text{ is isolated})\phi(y \text{ is isolated})},$$

which is at least 1, by the positive association of $\phi$. $\square$