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Hyperbolic Site Percolation

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ABSTRACT

Several results are presented for site percolation on quasi-transitive, planar graphs G with one end, when properly embedded in either the Euclidean or hyperbolic plane. If (G_1, G_2) is a matching pair derived from some quasi-transitive mosaic M, then $p_u(G_1) + p_c(G_2) = 1$, where p_c is the critical probability for the existence of an infinite cluster, and p_u is the critical value for the existence of a *unique* such cluster. This fulfils and extends to the hyperbolic plane an observation of Sykes and Essam (1964), and it extends to quasi-transitive site models a theorem of Benjamini and Schramm (Thm. 3.8, *Journal of the American Mathematical Society* 14 (2001): 487–507) for transitive bond percolation. It follows that $p_u(G) + p_c(G_*) = p_u(G_*) + p_c(G) = 1$, where G_* denotes the matching graph of G. In particular, $p_u(G) + p_c(G) \geq 1$ and hence, when G is amenable we have $p_c(G) = p_u(G) \geq \frac{1}{2}$. When combined with the main result of the companion paper by the same authors (*Random Structures & Algorithms* (2024)), we obtain for transitive G that the strict inequality $p_u(G) + p_c(G) > 1$ holds if and only if G is not a triangulation. A key technique is a method for expressing a planar site percolation process on a matching pair in terms of a dependent bond process on the corresponding dual pair of graphs. Amongst other matters, the results reported here answer positively two conjectures of Benjamini and Schramm (Conj. 7, 8, *Electronic Communications in Probability* 1 (1996): 71–82) in the case of quasi-transitive graphs.

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1 | Introduction and Results

1.1 | Percolation on Planar Graphs

Percolation was introduced in 1957 by Broadbent and Hammersley [1] as a model for the spread of fluid through a random medium. Percolation provides a natural mathematical setting for such topics as the study of disordered materials, magnetization, and the spread of disease. See [2–4] for recent accounts of the theory. We consider here site percolation on a graph $G = (V, E)$, assumed to be infinite, locally finite, connected, and planar. The current work has two linked objectives.

Our major objective is to study the relationship between the percolation critical point p_c and the critical point p_u marking the existence of a *unique* infinite cluster. More specifically, we establish the formula $p_u^{\text{site}}(G_1) + p_c^{\text{site}}(G_2) = 1$ for a matching pair (G_1, G_2) of graphs arising from a quasi-transitive mosaic, appropriately embedded in either the Euclidean or hyperbolic plane. See Section 1.2.

Setting $(G_1, G_2) = (G, G_*)$ above, with G_* the matching graph of G, we obtain

$$p_u^{\text{site}}(G) + p_c^{\text{site}}(G_*) = p_u^{\text{site}}(G_*) + p_c^{\text{site}}(G) = 1.$$

It follows that $p_u^{site}(G) + p_c^{site}(G) > 1$ if and only if the strict inequality $p_c^{site}(G_*) < p_c^{site}(G)$ holds. Necessary and sufficient conditions for the last inequality are established in [5] (for transitive graphs) and [6] (for quasi-transitive graphs). For transitive G, this implies that $p_u^{site}(G) + p_c^{site}(G) > 1$ if and only if G is not a triangulation.

Our second objective, which is achieved in the process of proving the above formula, is to validate Conjectures 7 and 8 of Benjamini and Schramm [7] concerning the existence of infinitely many infinite clusters. Details of these conjectures are found in Section 1.3.

The organization of the article is presented in Section 1.4.

1.2 | Critical Points of Matching Pairs

Since loops and multiple edges have no effect on the existence of infinite clusters in site percolation, the graphs considered in this article are generally assumed to be *simple* (whereas their dual graphs may be non-simple). The main results proved in this paper are as follows (see Sections 2.1 and 2.2 for explanations of the standard notation used here).

The word 'transitive' shall mean 'vertex-transitive' throughout this work. We denote by

$\mathscr{G}$: all infinite, locally finite, planar, 2-connected, simple graphs,

$\mathscr{T}$: the subset of $\mathscr{G}$ containing all such transitive graphs,

$\mathscr{Q}$: the subset of $\mathscr{G}$ containing all such quasi-transitive graphs.

Since the work reported here concerns matching and dual graphs, the graphs in $\mathcal{G}$ will be considered in their plane embeddings. The most interesting such graphs turn out to be those with one end. We shall recall in Section 3.1 that one-ended graphs in $\mathcal{T}$ have unique proper embeddings in the Euclidean/hyperbolic plane up to homeomorphism, and hence their matching and dual graphs are uniquely defined. The situation is more complicated for one-ended graphs in $\mathcal{Q}$, in which case we fix a plane embedding of $G \in \mathcal{Q}$ for which the dual graph G^+ is quasi-transitive. Such an embedding is called *canonical*; if G has connectivity 2, a canonical embedding need not be unique (even up to homeomorphism), but its existence is guaranteed by Theorem 3.1(c).

Matching pairs of graphs were introduced by Sykes and Essam [8] and explored further by Kesten [9]. Let $M \in \mathcal{Q}$ be one-ended and canonically embedded in the plane (we call M a *mosaic* following the earlier literature). Let $\mathcal{F}_4 = \mathcal{F}_4(M)$ be the set of faces of M bounded by n-cycles with $n \geq 4$, and let $\mathcal{F}_4 = F_1 \cup F_2$ be an arbitrary quasi-transitive partition of $\mathcal{F}_4$. The graph G_i is obtained from M by adding all diagonals to all faces in F_i. The pair (G_1, G_2) is called a *matching pair*. The *matching graph* G_* of a one-ended graph $G \in \mathcal{Q}$ is obtained by adding all diagonals to all faces in $\mathcal{F}_4(G)$. Thus, (G, G_*) is an instance of a matching pair. Two examples of matching pairs are given in Figure 1.

The notation p_u denotes the critical value for the existence of a *unique* infinite cluster. Further notation and background for percolation is deferred to Section 2.2.

Theorem 1.1.

(a) Let (G_1, G_2) be a matching pair derived from the mosaic $M \in \mathcal{Q}$. We have that

$$p_u^{site}(G_1) + p_c^{site}(G_2) = 1. \tag{1}$$

(b) Let $G \in \mathcal{Q}$ be one-ended. Then

$$p_u^{site}(G) + p_c^{site}(G) \geq 1. \tag{2}$$

If G is transitive, equality holds in (2) if and only if G is a triangulation.

In the context of (1), Sykes and Essam [8, eq. (7.3)] presented motivation for the exact formula

$$p_c^{site}(G_1) + p_c^{site}(G_2) = 1, \tag{3}$$

and this has been verified in a number of cases when G is amenable (see [10]). This formula does not hold for non-amenable graphs. Equation (1) appears without proof in [11, eq. (4)] for a restricted class of graphs.

Remark 1.1. (Strict inequality). Equation (2) follows from (1) with $(G_1, G_2) = (G, G_*)$, by the inequality $p_c^{site}(G) \geq p_c^{site}(G_*)$. This weak inequality holds trivially since G is a subgraph of G_*. The corresponding strict inequality $p_c^{site}(G) > p_c^{site}(G_*)$ is investigated in the companion papers [5, 6], where necessary and sufficient conditions are presented. By (1),

$$p_u^{site}(G) - p_u^{site}(G_*) = p_c^{site}(G) - p_c^{site}(G_*) \geq 0,$$

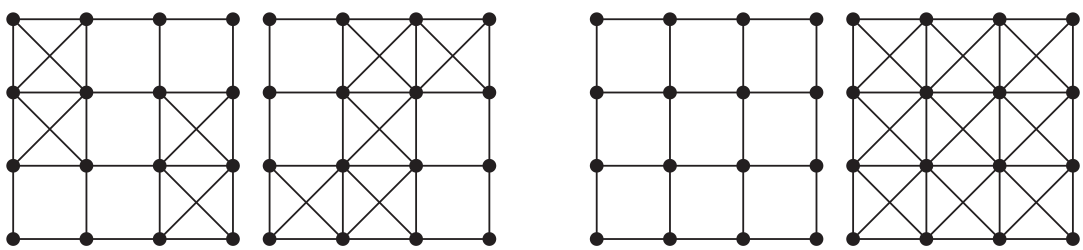

FIGURE 1 | Two matching pairs derived from the square lattice $\mathbb{Z}^2$. Each 3×3 grid is repeated periodically about $\mathbb{Z}^2$. The pair on the right generates $\mathbb{Z}^2$ and its matching graph.

so that strict inequality for p_c^{site} is equivalent to strict inequality for p_u^{site}.

Remark 1.2. (Canonical embeddings). When G has connectivity 2, it may possess more than one canonical embedding; by Theorem 1.1, $p_c^{site}(G_*)$ and $p_u^{site}(G_*)$ are independent of the choice of canonical embedding. This may be seen directly by observing that, in situations where there is a choice of embedding, the two-point connectivity functions are equal.

Remark 1.3. (Amenability). If $G \in \mathcal{Q}$ is one-ended and in addition amenable, by the uniqueness of the infinite cluster [12, 13], we have $p_c^{site}(G) = p_u^{site}(G)$; in this case, $p_c^{site}(G) \geq \frac{1}{2}$ by (2). If G is transitive, we have $p_c^{site}(G) = \frac{1}{2}$ if and only if G is the usual amenable, triangular lattice.

The dual graph of a plane graph G is denoted G^+.

Remark 1.4. (Bond percolation). Theorem 1.1 may be compared with the corresponding results for bond percolation. It is proved in [14, Thm. 3.8] that

$$p_c^{bond}(G) + p_u^{bond}(G^+) = 1$$

for any non-amenable, *transitive* $G \in \mathcal{T}$. If, instead, $G \in \mathcal{T}$ is amenable, it is standard that $p_u^{bond}(G^+) = p_c^{bond}(G^+) = 1 - p_c^{bond}(G)$. These facts are extended to quasi-transitive graphs in [4, Thm. 8.31]. We make use of some elements of [4, 14] here, while studying the more general site percolation directly via the concept of pairs of matching graphs.

1.3 | Existence of Infinitely Many Infinite Clusters

A number of problems for percolation on non-amenable graphs were formulated by Benjamini and Schramm in their influential paper [7], including the following two conjectures.

Conjecture 1.1. ([7, Conj. 7]). Consider site percolation on an infinite, connected, planar graph G with minimal degree at least 7. Then, for any $p \in (p_c^{site}, 1 - p_c^{site})$, we have $\mathbb{P}_p(N = \infty) = 1$. Moreover, it is the case that $p_c^{site} < \frac{1}{2}$, so the above interval is invariably non-empty.

It was proved in [15, Thm. 2] that $p_c^{site} < \frac{1}{2}$ for planar graphs with vertex-degrees at least 7.

Conjecture 1.2. ([7, Conj. 8]). Consider site percolation on a planar graph G satisfying $\mathbb{P}_{\frac{1}{2}}(N \geq 1) = 1$. Then $\mathbb{P}_{\frac{1}{2}}(N = \infty) = 1$.

Percolation in the hyperbolic plane was later studied by Benjamini and Schramm [14]. In the current paper, we extend certain results of [14] to amenable planar graphs and to site percolation, and we confirm Conjectures 1.1 and 1.2 for all planar, quasi-transitive graphs.

Conjectures 1.1 and 1.2 were verified in [16] when G is a regular triangular tiling (or 'triangulation') of the hyperbolic plane $\mathcal{H}$ for which each vertex has degree at least 7. A significant property of a triangulation is that its matching graph is the same as the original graph.

The next two theorems establish Conjectures 1.1 and 1.2 for planar, quasi-transitive graphs.

Theorem 1.2. Consider site percolation on a graph $G \in \mathcal{Q}$, each vertex of which has degree 7 or more.

(a) For every $p \in (p_c^{site}, 1 - p_c^{site})$, there exist, $\mathbb{P}_p$-a.s., infinitely many infinite 1-clusters and infinitely many infinite 0-clusters.

(b) For every $p \in [0, 1]$, there exists, $\mathbb{P}_p$-a.s., at least one infinite cluster that is either a 1-cluster or a 0-cluster.

Theorem 1.3. Consider site percolation on $G \in \mathcal{Q}$, and assume that $\mathbb{P}_{\frac{1}{2}}(N \geq 1) = 1$. Then, $\mathbb{P}_{\frac{1}{2}}$-a.s., there exist infinitely many infinite 1-clusters and infinitely many infinite 0-clusters.

The approach to establishing Conjectures 1.1 and 1.2 is to classify $\mathcal{Q}$ according to amenability and the number of ends, and then prove these conjectures for each such subclass of graphs. We recall the following well-known theorem.

Theorem 1.4. ([17, 18, Prop. 2.1]). A graph G that is infinite, connected, locally finite, and quasi-transitive has either one or two or infinitely many ends. If it has two ends, then it is amenable. If it has infinitely many ends, then it is non-amenable.

Let $G \in \mathcal{Q}$. By Theorem 1.4, only the following cases may occur.

(i) G is amenable and one-ended. This case includes the square lattice, for which percolation has been studied extensively; see, for example, [3, 9].

(ii) G is non-amenable and one-ended. It is proved in [14] that $p_c^{site} < p_u^{site}$ and $p_c^{bond} < p_u^{bond}$ for this case.

(iii) G has two ends, in which case there is no percolation phase transition of interest.

(iv) G has infinitely many ends.

We shall study percolation on each class of graphs listed above. Matching graphs and dual graphs will play important roles in our analysis.

1.4 | Organization of Material

Section 2 is devoted to basic notation for graphs and percolation. In Section 3, we review certain known results that will be used to prove the main results of Section 1.2. It is explained in Section 4 how a site percolation process on a planar graph may be expressed in terms of a dependent bond process on the same graph; this allows a connection between site percolation on the matching graph and bond percolation on the dual graph. We prove Theorem 1.1(a) for amenable graphs in Section 5, and for non-amenable graphs in Section 6. Theorem 1.2 is proved in Section 7, and Theorem 1.3 in Section 8.

2 | Notation

2.1 | Graphical Notation

Let $\mathrm{Aut}(G)$ be the automorphism group of the graph $G = (V, E)$. A graph G is called *vertex-transitive*, or simply *transitive*, if all the vertices lie in the same orbit under the action of $\mathrm{Aut}(G)$. The graph G is called *quasi-transitive* if the action of $\mathrm{Aut}(G)$ on V has only finitely many orbits. It is called *locally finite* if all vertex-degrees are finite. An edge with endpoints u, v is denoted $\langle u, v\rangle$, in which case we call u and v *adjacent* and we write $u \sim v$. The graph-distance $d_G(u, v)$ between vertices u, v is the minimal number of edges in a path from u to v.

A graph G is *planar* if it can be embedded in the plane $\mathbb{R}^2$ in such a way that its edges intersect only at their endpoints; a planar embedding of such G is called a *plane* graph. A *face* of a plane graph G is an (arc-)connected component of the complement $\mathbb{R}^2 \setminus G$. Note that faces are open sets, and may be either bounded or unbounded. With a face F, we associate the set of vertices and edges in its boundary. The *size* of a face is the number of edges in its boundary. While it may be helpful to think of a face as being bounded by a cycle of G, the reality can be more complicated in that faces are not invariably simply connected (if G is disconnected) and their boundaries are not generally self-avoiding cycles or paths (if G is not 2-connected). A plane graph G is called a *triangulation* it every face is bounded by a 3-cycle.

A manifold M is called *plane* if it is a surface and, for every self-avoiding cycle π of M, $M \setminus \pi$ has exactly two connected components. When a graph is drawn in a plane manifold M, the terms embedding and face mean the same as when embedded in the Euclidean plane. We say that an embedded graph $G \subset M$ is *properly embedded* if every compact subset of M contains only finitely many vertices of G and intersects only finitely many edges. Henceforth, all embeddings will be assumed to be proper. The term *plane* shall mean either the Euclidean plane or the hyperbolic plane, and each may be denoted $\mathcal{H}$ when appropriate.

A *cycle* (or *n-cycle*) C of a simple graph $G = (V, E)$ is a sequence $v_0, v_1, \dots, v_{n+1} = v_0$ of vertices v_i such that $n \ge 3$, $e_i := \langle v_i, v_{i+1}\rangle$ satisfies $e_i \in E$ for $i = 0, 1, \dots, n$, and $v_0, v_1, \dots, v_n$ are distinct. Let G be a plane graph, properly embedded in $\mathcal{H}$. In this case we write C° for the bounded component of $\mathbb{R}^2 \setminus C$, and $\overline{C}$ for the closure of C°. The 'matching graph' G_* is obtained from G by adding all possible diagonals to every face of G. That is, let F be such a face, and let ∂F be the set of vertices lying in the boundary of F. We augment G by adding edges between any distinct pair $x, y \in V$ such that (i) there exists a face F such that $x, y \in \partial F$ and (ii) $\langle x, y\rangle \notin E$. We write D for the set of diagonals, so that $G_* = (V, E \cup D)$. We recall from [19, Thm. 3] (see Remark 3.1(d)) that, for a 2-connected graph G, every face is bounded by either a cycle or a doubly-infinite path.

Next we define a matching pair. Let $M \in \mathcal{Q}$ be one-ended (we follow the earlier literature by calling M a *mosaic* in this context). By the forthcoming Remark 3.1(d), M has an embedding in the plane such that the dual graph M^+ and the matching graph M_* are quasi-transitive, and furthermore every face of M is bounded by a cycle. Let $\mathcal{F}_4 = \mathcal{F}_4(M)$ be the set of faces of M bounded by n-cycles with $n \ge 4$, and let $\mathcal{F}_4 = F_1 \cup F_2$ be a partition of $\mathcal{F}_4$. The graph G_i is obtained from M by adding all diagonals to all faces in F_i, and we assume that $\mathrm{Aut}(M)$ has some subgroup Γ that acts quasi-transitively on each G_i. The pair (G_1, G_2) is said to be a *matching pair* derived from M.

The graph G is called *amenable* if its Cheeger constant satisfies

$$\inf_{K \subseteq V,\ |K| < \infty} \frac{|\Delta K|}{|K|} = 0 \tag{4}$$

where ΔK is the subset of E containing edges with exactly one endpoint in K. If the left side of (4) is strictly positive, the graph G is called *non-amenable*.

Each $G \in \mathcal{T}$ is quasi-isometric with one and only one of the following spaces: $\mathbb{Z}$, the 3-regular tree, the Euclidean plane, and the hyperbolic plane; see [14, 18]. See [20, 21] for background on hyperbolic geometry.

Recall that the number of ends of a connected graph is the supremum over its finite subgraphs F of the number of infinite components that remain after removing F, and recall Theorem 1.4. The number of ends of a graph is highly relevant to properties of statistical mechanical models on the graph; see [22, 23], for example, for discussions of the relevance of the number of ends to the number and speed of self-avoiding walks.

2.2 | Percolation Notation

Let $G = (V, E)$ be a connected, simple graph with bounded vertex-degrees. A *site percolation* configuration on G is an assignment $\omega \in \Omega_V := \{0, 1\}^V$ to each vertex of either state 0 or state 1. A cluster in ω is a maximal connected set of vertices in which each vertex has the same state. A cluster may be a 0-cluster or a 1-cluster depending on the common state of its vertices, and it may be finite or infinite. We say that 'percolation (or 1-percolation) occurs' in ω if there exists an infinite 1-cluster in ω. For $\omega \in \Omega_V$, we write $1 - \omega$ for the configuration with open/closed inverted.

A *bond percolation* configuration $\omega \in \Omega_E := \{0, 1\}^E$ is an assignment to each edge in G of either state 0 or state 1. A bond percolation model may be considered as a site percolation model on the so-called *covering graph* (or *line graph*) $\widetilde{G}$ of G. Therefore, we may use the term 1-cluster (respectively, 0-cluster) for a maximal connected set of edges with state 1 (respectively, state 0) in a bond configuration. The *size* of a cluster in site/bond percolation is the number of its vertices.

We call a vertex or an edge *open* if it has state 1, and *closed* otherwise. Let μ be a probability measure on Ω_V endowed with the product σ-field. The corresponding site model is the probability space (Ω_V, μ), with a similar definition for a bond model (Ω_E, μ). The central questions in percolation theory concern the existence and multiplicity of infinite clusters viewed as functions of μ.

A percolation model (Ω, μ) is called *invariant* if μ is invariant under the action of $\mathrm{Aut}(G)$. An invariant measure is called *ergodic* if there exists an automorphism subgroup Γ acting quasi-transitively on G such that $\mu(A) \in \{0,1\}$ for any Γ-invariant event A. See, for example, [4, Prop. 7.3]. It is standard that the product measure $\mathbb{P}_p$ is ergodic if G is infinite and quasi-transitive.

Consider percolation on a graph $G = (V, E)$. A site or bond configuration ω induces open and closed subgraphs of G in the usual way, and we write N $(= N_G(\omega))$ for the number of infinite 1-clusters, and $\overline{N}$ $(= \overline{N}_G(\omega))$ for the number of infinite 0-clusters. For site percolation on a graph G, we write N_*, $\overline{N}_*$ for the corresponding quantities on the matching graph G_*. A configuration is in one–one correspondence with the set of elements (vertices or edges, as appropriate) that are open in the configuration.

Let $p \in [0,1]$. We endow Ω_V with the product measure $\mathbb{P}_p$ with density p. For $v \in V$, let $\theta_v(p)$ be the probability that v lies in an infinite open cluster. It is standard that there exists $p_c^{\mathrm{site}}(G) \in (0,1]$ such that

$$\text{for } v \in V,\ \theta_v(p) \begin{cases} = 0 & \text{if } p < p_c^{\mathrm{site}}(G), \\ > 0 & \text{if } p > p_c^{\mathrm{site}}(G), \end{cases}$$

and $p_c^{\mathrm{site}}(G)$ is called the *(site) critical probability* of G.

More generally, consider (either bond or site) percolation on a graph G with probability measure $\mathbb{P}_p$. The corresponding critical points may be expressed as follows:

$$p_c^{\mathrm{site}}(G) := \inf\{p \in [0,1] : \mathbb{P}_p(N \geq 1) = 1 \text{ for site percolation}\},$$
$$p_c^{\mathrm{bond}}(G) := \inf\{p \in [0,1] : \mathbb{P}_p(N \geq 1) = 1 \text{ for bond percolation}\},$$

and

$$p_u^{\mathrm{site}}(G) := \inf\{p \in [0,1] : \mathbb{P}_p(N = 1) = 1 \text{ for site percolation}\},$$
$$p_u^{\mathrm{bond}}(G) := \inf\{p \in [0,1] : \mathbb{P}_p(N = 1) = 1 \text{ for bond percolation}\}.$$

By the Kolmogorov zero–one law, $\mathbb{P}_p(N \geq 1)$ equals either 0 or 1.

The notation p_c (respectively, p_u) shall always mean the critical probability p_c^{site} (respectively, p_u^{site}) of the site model. For background and notation concerning percolation theory, the reader is referred to the book [3].

3 | Background

We review certain known results that will be used in the proofs of our main results.

3.1 | Embeddings of One-Ended Planar Graphs

We say that the 2-sphere, the Euclidean plane, and the hyperbolic plane constitute the *natural geometries* (see, e.g., Babai [18, sect. 3.1]). The natural geometries are two-dimensional Riemannian manifolds. An *Archimedean tiling* of a two-dimensional Riemannian manifold is a tiling by regular polygons such that the group of isometries of the tiling acts transitively on the vertices of the tiling. An infinite, one-ended, transitive planar graph can be characterized as a tiling of either the Euclidean plane or the hyperbolic plane.

An *embedding* of a graph $G = (V, E)$ (with underlying 1-complex denoted $|G|$) in a surface M is a continuous map $\phi : |G| \to M$ such that the induced map $|G| \to \phi(|G|)$ is a homeomorphism. An embedding ϕ is called *cellular* if $M \setminus \phi(G)$ is a disjoint union of spaces homeomorphic to an open disc (see [24] and [25, sect. 3.2]).

We shall consider embeddings of planar graphs in either the Euclidean or hyperbolic plane, and we use the notation $\mathcal{H}$ to denote either of these, as appropriate for the context.

Theorem 3.1.

(a) [18, Thms 3.1, 4.2]: If $G \in \mathcal{T}$ is one-ended, then G may be embedded in $\mathcal{H}$ as an Archimedean tiling, and all automorphisms of G extend to isometries of $\mathcal{H}$. If $G \in \mathcal{Q}$ is one-ended and 3-connected, then G may be embedded in $\mathcal{H}$ such that all automorphisms of G extend to isometries of $\mathcal{H}$.

(b) [24, p. 42]: Let G be a 3-connected graph, cellularly embedded in $\mathcal{H}$ such that all faces are of finite size. Then G is uniquely embeddable in the sense that for any two cellular embeddings $\phi_1 : G \to S_1$, $\phi_2 : G \to S_2$ into planar surfaces S_1, S_2, there is a homeomorphism $\tau : S_1 \to S_2$ such that $\phi_2 = \tau\phi_1$.

(c) [4, Thm. 8.25 and Proof, pp. 288, 298]: If $G = (V, E) \in \mathcal{Q}$ is one-ended, there exists some embedding of G in $\mathcal{H}$ such that the edges coincide with geodesics, the dual graph G^+ is quasi-transitive, and all automorphisms of G extend to isometries of $\mathcal{H}$. Such an embedding is called *canonical*.

(d) [26]: The automorphism group $\mathrm{Aut}(G)$ of a quasi-transitive graph G with quadratic growth contains a subgroup isomorphic to $\mathbb{Z}^2$ that acts quasi-transitively on G.

Remark 3.1. Some known facts concerning embeddings follow.

(a) [27, Props 2.2, 2.2]: All one-ended, transitive, planar graphs are 3-connected, and all proper embeddings of a one-ended, quasi-transitive, planar graph have only finite faces.

(b) By Theorem 3.1(b), any one-ended $G \in \mathcal{T}$ has a unique proper cellular embedding in $\mathcal{H}$ up to homeomorphism. Hence, the matching and dual graphs of G are independent of the embedding.

(c) The conclusion of part (b) holds for any one-ended, 3-connected $G \in \mathcal{Q}$.

(d) For a one-ended $G \in \mathcal{Q}$, we fix a canonical embedding (in the sense of Theorem 3.1(c)). With this given, the dual graph G^+ and the matching graph G_* are quasi-transitive, and furthermore (by [19, Thm. 3]) the boundary of every face is a cycle of G.

Remark 3.2. (Proper embedding). Theorem 3.1(a) implies in particular that every one-ended $G \in \mathcal{T}$ may be properly embedded in its natural geometry. Such an embedding is called topologically locally finite (TLF) by Renault [28, Prop. 5.1, 29]. For a related discussion in the case of non-amenable graphs, see [14, Prop. 2.1].

Remark 3.3. (Connectivity). Graphs with connectivity 1 have been excluded from membership of $\mathcal{G}$ (and therefore from $\mathcal{T}$ and $\mathcal{Q}$ also). Percolation on such graphs has little interest since any finite dangling ends may be removed without changing the existence of an infinite cluster. Moreover, let F be a face of a mosaic M, such that F contains some dangling end D. If (G_1, G_2) is a matching pair derived from M, the critical values $p_c(G_i)$ are unchanged if D is deleted.

The representation of transitive, planar graphs as tilings of natural geometries enables the development of universal techniques to study statistical mechanical models on all such graphs; see, for example, the study [22] of a universal lower bound for connective constants on infinite, connected, transitive, planar, cubic graphs.

3.2 | Percolation

We assume throughout this subsection that the graph G is infinite, connected, and locally finite.

Lemma 3.1. ([30, Cor. 1.2, 31]). Let G be quasi-transitive, and consider either site or bond percolation on G. Let $0 < p_1 < p_2 \le 1$, and assume that $\mathbb{P}_{p_1}(N = 1) = 1$. Then $\mathbb{P}_{p_2}(N = 1) = 1$.

Definition 3.1. Let $G = (V, E)$ be a graph. Given $\omega \in \Omega_V$ and a vertex $v \in V$, write $\Pi_v \omega = \omega \cup \{v\}$ (which is to say that v is declared open). For $A \subseteq \Omega_V$, we write $\Pi_v A = \{\Pi_v \omega : \omega \in A\}$. A site percolation process (Ω_V, μ) on G is called insertion-tolerant if $\mu(\Pi_v A) > 0$ for every $v \in V$ and every event $A \subseteq \Omega_V$ satisfying $\mu(A) > 0$.

A site percolation is called deletion-tolerant if $\mu(\Pi_{\neg v} A) > 0$ whenever $v \in V$ and $\mu(A) > 0$, where $\Pi_{\neg v}\omega = \omega \setminus \{v\}$ for $\omega \in \Omega_V$, and $\Pi_{\neg v} A = \{\Pi_{\neg v}\omega : \omega \in A\}$.

Similar definitions apply to bond percolation. We shall encounter weaker definitions in Section 3.3.

Lemma 3.2. ([4, Thm. 7.8, 32, Thm. 8.1]). Let $G = (V, E)$ be a connected, locally finite, quasi-transitive graph, and let (Ω, μ) be an invariant (site or bond) percolation on G. Assume either or both of the following two conditions hold:

(a) (Ω, μ) is insertion-tolerant,

(b) G is a non-amenable planar graph with one end.

Then $\mu(N \in \{0, 1, \infty\}) = 1$. If μ is ergodic, N is μ-a.s. constant.

The sufficiency of (a) is proved in [4, Thm. 7.8] for transitive graphs, and the same proof is valid for quasi-transitive graphs. The sufficiency of (b) is proved in [32, Thm. 8.1].

3.3 | Planar Duality

Let $G = (V, E)$ be a plane graph, and write $\mathcal{F}$ for the set of its faces. The dual graph $G^+ = (V^+, E^+)$ is defined as follows. The sets V^+ and $\mathcal{F}$ are in one–one correspondence, written $v_f \leftrightarrow f$. Two vertices $v_f, v_g \in V^+$ are joined by $n_{f,g}$ parallel edges where $n_{f,g}$ is the number of edges of E common to the faces $f, g \in \mathcal{F}$. Thus, E^+ and E are in one–one correspondence, written $e^+ \leftrightarrow e$.

For a bond configuration $\omega \in \Omega_E$, we define the dual configuration $\omega^+ \in \Omega_{E^+}$ by: for each dual pair $(e, e^+) \in E \times E^+$ of edges, we have

$$\omega(e) + \omega^+(e^+) = 1. \tag{5}$$

In the following, (Ω_E, μ) is a bond percolation model on $G = (V, E)$. Similar definitions apply to site percolation.

Definition 3.2. A probability measure μ is called weakly insertion-tolerant if there exists a function $f : E \times \Omega_E \to \Omega_E$ such that

(a) for all e and all $\omega \in \Omega_E$, we have $\omega \cup \{e\} \subseteq f(e, \omega)$,

(b) for all e and all ω, the difference $f(e, \omega) \setminus [\omega \cup \{e\}]$ is finite, and

(c) for all e and each event A satisfying $\mu(A) > 0$, the image of A under $f(e, \cdot)$ is an event of strictly positive probability.

Definition 3.3. A probability measure μ is called weakly deletion-tolerant if there exists a function $h : E \times \Omega_E \to \Omega_E$ such that

(a) for all e and all $\omega \in \Omega_E$, we have $\omega \setminus \{e\} \supseteq h(e, \omega)$,

(b) for all e and all ω, the difference $[\omega \setminus \{e\}] \setminus h(e, \omega)$ is finite, and

(c) for all e and each event A satisfying $\mu(A) > 0$, the image of A under $h(e, \cdot)$ is an event of strictly positive probability.

Lemma 3.3. ([4, Thm. 8.30]). Let $G = (V, E) \in \mathcal{Q}$ be non-amenable and one-ended, and consider G embedded canonically in the plane (such an embedding exists by Theorem 3.1(c)). Let (Ω_E, μ) be an invariant, ergodic, bond percolation on G, assumed to be both weakly insertion-tolerant and weakly deletion-tolerant. For $\omega \in \Omega_E$, let $N(\omega)$ be the number of infinite open components of ω, and $N^+(\omega)$ the number of infinite open components of the dual process ω^+ given in (5). Then

$$\mu\left((N, N^+) \in \{(0,1), (1,0), (\infty, \infty)\}\right) = 1.$$

3.4 | Graphs With Two or More Ends

We summarise here the main results for critical percolation probabilities on multiply-ended graphs.

Theorem 3.2. ([33, 34]). Let $G \in \mathcal{Q}$ have two ends. The critical percolation probabilities satisfy

$$p_c^{\text{bond}}(G) = p_c^{\text{site}}(G) = p_u^{\text{bond}}(G) = p_u^{\text{site}}(G) = 1.$$

Theorem 3.3. Let $G \in \mathcal{Q}$ have infinitely many ends. Then

$$p_c^{\text{bond}}(G) \le p_c^{\text{site}}(G) < p_u^{\text{bond}}(G) = p_u^{\text{site}}(G) = 1.$$

The standard inequality $p_c^{\text{bond}} \le p_c^{\text{site}}$ holds for all graphs, and was stated in [35]. The corresponding strict inequality was explored in [36, Thm. 2] for bridgeless, quasi-transitive graphs. The equalities $p_u^{\text{bond}} = p_u^{\text{site}} = 1$ were proved for transitive graphs in [34, eq. (3.7)] (see also [33]), and feature in [4, Exer. 7.9] for quasi-transitive graphs. The inequality $p_c^{\text{site}} < 1$ for non-amenable graphs was given in [7, Thm. 2].

3.5 | FKG Inequality

For completeness, we state the well-known FKG inequality. See, for example [3, sect. 2.2], for further details.

Theorem 3.4. (FKG inequality [37, 38]). Let μ be a strictly positive probability measure on Ω_V satisfying the FKG lattice condition:

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \ge \mu(\omega_1)\mu(\omega_2), \qquad \omega_1, \omega_2 \in \{0,1\}^V. \quad (6)$$

For any increasing events $A, B \subseteq \{0,1\}^V$, we have that $\mu(A \cap B) \ge \mu(A)\mu(B)$.

4 | Planar Site Percolation as a Bond Model

Let $M = (V, E) \in \mathcal{Q}$ be a mosaic, and let (G_1, G_2) be a matching pair derived from M according to the partition $\mathcal{F}_4(M) = F_1 \cup F_2$. If $F_i \ne \emptyset$, then G_i is non-planar. This is an impediment to consideration of the dual graph of G_i, which in turn is overcome by the introduction of so-called facial sites.

Let $\mathcal{F} = \mathcal{F}(M)$ be the set of faces of M (following [9], we include triangular faces). The triangular faces of $\mathcal{F}$ do not appear in $F_1 \cup F_2 = \mathcal{F}_4$, but we allocate each such face arbitrarily to either F_1 of F_2 (for concreteness, we add them all to F_1). One may replace the mosaic M by the triangulation $\widehat{M}$ obtained by placing a *facial site* $\phi(F)$ inside each face $F \in \mathcal{F}$, and joining $\phi(F)$ to each vertex in the boundary of F (see [9, sect. 2.3] and [5, sect. 4.2].).

When considering site percolation on M (respectively, M_*), one declares the facial sites of $\widehat{M}$ to be invariably closed (respectively, open). Site percolation on G_i is equivalent to site percolation on $\widehat{M}$ subject to:

$$\text{a facial site } \phi(F) \text{ is declared open if } F \in F_i \text{ and closed if } F \in \mathscr{F} \setminus F_i. \quad (7)$$

Note that, if F is a triangular face, the state of $\phi(F)$ is independent of the connectivity of its other vertices.

The *facial graph* $\widehat{G}_i$ is obtained by adding to M the facial sites of F_i only, together with their incident edges. We write $\widehat{G}_i = (V_i, E_i) := (V \cup \Phi_i, E \cup \eta_i)$ where Φ_i is the set of facial sites of G_i and η_i is the set of edges incident to facial sites. We shall consider two site percolation processes, namely, percolation of open sites on $\widehat{G}_1$ and of closed sites on $\widehat{G}_2$ (subject to (7)). To this end, for $\omega \in \Omega_V$, let ω_1 (respectively, ω_2) be the site configuration on $\widehat{G}_1$ (respectively, $\widehat{G}_2$) that agrees with ω on V and is *open* on Φ_1 (respectively, *closed* on Φ_2).

Given $\omega \in \Omega_V$, we construct a bond configuration $\beta_{\omega_1} \in \Omega_{E \cup \eta_1}$ by

$$\beta_{\omega_1}(e) = \begin{cases} 1 & \text{if } \omega_1(u) = \omega_1(v) = 1 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

where $e = \langle u, v \rangle \in E \cup \eta_1$. Let $\beta_{\omega_1}^+ := 1 - \beta_{\omega_1}$ be the corresponding dual configuration on the dual graph $\widehat{G}_1^+ = (V_1^+, E_1^+)$ of $\widehat{G}_1$ as in (5). Let $\widehat{G}_1(\beta_{\omega_1})$ (respectively, $\widehat{G}_1^+(\beta_{\omega_1}^+)$) be the graph with vertex-set V_1 (respectively, V_1^+) endowed with the open edges of β_{ω_1} (respectively, $\beta_{\omega_1}^+$). Note that, if ω has law $\mathbb{P}_p$, then the law of β_{ω_1} is one-dependent. We may identify the vector β_{ω_1} with the set of its open edges.

Lemma 4.1. Suppose $\omega \in \Omega_V$ has law $\mathbb{P}_p$ where $p \in (0,1)$. The law μ of β_{ω_1} is weakly deletion-tolerant and weakly insertion-tolerant. Moreover, μ is ergodic.

Proof. Let $e = \langle u, v \rangle \in E \cup \eta_1$ and $\omega \in \Omega_V$. For $w \in V$, let D_w be the set of edges of $\widehat{G}_1$ of the form $\langle w, x \rangle$ with $\omega(x) = 1$. Select an endvertex, u say, of e that is not a facial site (such a vertex always exists), and define

$$f(e, \beta_{\omega_1}) = \beta_{\omega_1} \cup (D_u \cup D_v \cup \{e\}),$$
$$h(e, \beta_{\omega_1}) = \beta_{\omega_1} \setminus (D_u \cup \{e\}).$$

The edge-configuration $f(e, \beta_{\omega_1})$ (respectively, $h(e, \beta_{\omega_1})$) is that obtained by setting u and v to be open (respectively, u to be closed). With these functions f, h, the conditions of Definitions 3.2 and 3.3 hold since G is locally finite. The ergodicity holds by the assumed quasi-transitivity of G_1 and the fact that $\mathbb{P}_p$ is a product measure (see the comment in Section 2.2). □

For $\omega \in \Omega_V$, let $\widehat{G}_1(\omega)$ be the subgraph of $\widehat{G}_1$ induced by the set of ω_1-open vertices (that is, the set of v with $\omega_1(v) = 1$), and define $\widehat{G}_2(\varpi)$ similarly in terms of closed vertices of ω_2 in $\widehat{G}_2$.

We make some notes concerning the relationship between $\widehat{G}_1(\omega)$, $\widehat{G}_2(\varpi)$, and $\widehat{G}_1^+(\beta_{\omega_1}^+)$, as illustrated in Figure 2. A *cutset* of a graph H is a subset of edges whose removal disconnects some previously connected component of H, and which is minimal with this property. Recall that a *face* of a plane graph $H = (V', E')$ is a connected component of $\mathcal{H} \setminus E'$. A face F can be bounded or unbounded, and it need not be simply connected. It has a boundary ΔF comprising edges of H; even when F is bounded and simply connected, the set ΔF of edges need not be cycle of H unless H is 2-connected.

Proposition 4.1. Let $M = (V, E) \in \mathcal{Q}$ be one-ended and embedded canonically in $\mathcal{H}$. Let $\omega \in \Omega_V$, and let F be a face (either bounded or unbounded) of $\widehat{G}_1(\omega)$.

(a) Let C be a cycle (respectively, doubly-infinite path) of $\widehat{G}_1(\omega)$. The set of edges of $\widehat{G}_1^+$ intersecting C forms a finite (respectively, infinite) cutset of $\widehat{G}_1^+$.

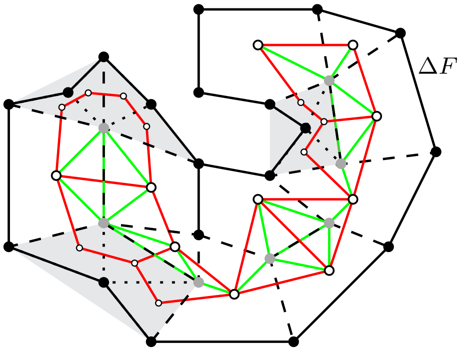

FIGURE 2 | An illustration of the one–one correspondence between $C_1(F)$ and $C_2(F)$ of Proposition 4.1. The black line is the boundary of the face F; the dashed lines are edges of M inside F; the dotted lines are edges of η_1. The shaded regions are faces of M that belong to F_1; the black points are open vertices; the grey points are closed vertices; the white points are dual vertices of $\widehat{G}_1$. The green graph is the 0-cluster $C_2(F)$ of $\widehat{G}_2(\omega)$ (i.e., the 1-cluster of $\widehat{G}_2(1-\omega)$) that corresponds to the red cluster $C_1(F)$ of $\widehat{G}_1^+(\beta_{\omega_1}^+)$.

(b) The set $F \cap V_1^+$ of dual vertices of $\widehat{G}_1$ inside F, together with the set of open edges of $\beta_{\omega_1}^+$ lying inside F, forms a non-empty, connected component $C_1(F)$ of $\widehat{G}_1^+(\beta_{\omega_1}^+)$.

(c) The set $F \cap (V \cup \Phi_2)$ of vertices of $\widehat{G}_2$ inside F forms a (possibly empty) 0-cluster $C_2(F)$ of $\widehat{G}_2(\overline{\omega})$.

(d) Either each of F, $C_1(F)$, $C_2(F)$ is bounded or each is unbounded.

Proof. (a) This is immediate by planar duality.

(b) Note first that every vertex w of M inside F satisfies $\omega(w) = 0$. Since F is bounded by a cycle of $\widehat{G}_1$, it is a non-empty, disjoint union $F = \bigcup_{i\in I} A_i$ of faces A_i of $\widehat{G}_1$ (more precisely, the two sides of the equality differ on a set of Lebesgue measure 0). Each A_i is bounded, and contains a (unique) dual vertex d_i. It is standard that the dual set $D = \{d_i : i \in I\}$ induces a connected graph $C_1(F)$ in F. Since no edge f of $C_1(F)$ intersects ΔF, we have $\beta_{\omega_1}^+(f) = 1$ for all such f.

(c) It can be the case that $F \cap (V \cup \Phi_2) = \emptyset$, in which case we take $C_2(F)$ to be the empty graph (this is the situation if and only if F is a triangular face of $\widehat{G}_1(\omega)$). Suppose henceforth that $F \cap (V \cup \Phi_2) \neq \emptyset$ and note as above that $\omega(w) = 0$ for every $w \in F \cap (V \cup \Phi_2)$. It is a standard property of matching pairs of graphs that $F \cap (V \cup \Phi_2)$ induces a connected subgraph $C_2(F)$ of $F \cap \widehat{G}_2$.

Parts (b) and (c) make use of two so-called 'standard' properties, full discussions of which are omitted here. It suffices to prove the 'standard' property of *matching* pairs, since the corresponding property for dual pairs then follows by passing to covering (or line) graphs (see, e.g., [9, sect. 2.6]). For matching pairs, an early reference is [8, app.], and a more detailed account is found in [9, sect. 3, app.] (see, in particular, Proposition A.1 of [9]). The latter assumes slightly more than here on the mosaic M, but the methods apply notwithstanding.

(d) When F is finite, so must be $C_1(F)$ and $C_2(F)$, since the embedding of M is proper. When F is infinite, the same holds of $C_1(F)$ and $C_2(F)$, since the faces of G are uniformly bounded. □

Recall the notation $N_G(\omega)$ from Section 2.2.

Proposition 4.2. Let $M = (V, E) \in \mathcal{Q}$ be one-ended and embedded canonically in $\mathcal{H}$, and let $\omega \in \Omega_V$. Then,

$$N_{G_1}(\omega) = N_{\widehat{G}_1}(\omega) = N_{\widehat{G}_1}(\beta_{\omega_1}),$$
$$N_{G_2}(1-\omega) = N_{\widehat{G}_2}(1-\omega) = N_{\widehat{G}_1^+}(\beta_{\omega_1}^+). \tag{9}$$

Proof. Equation (9) holds by the definition of β_{ω_1}, and from Proposition 4.1 on noting (for given ω) the one–one correspondence between infinite clusters of $\widehat{G}_2(1-\omega)$ and of $\widehat{G}_1^+(\beta_{\omega_1}^+)$. The facial site in any face of M is a surrogate for the diagonals of that face. □

Remark 4.1. (Conformality). It is classical that every bond percolation model may be phrased as a site model on the so-called covering (or line) graph (see, e.g., [3, p. 24]). While the converse is generally false, using definition (8) we obtain a one-dependent bond model from the site model on the same graph; furthermore, the connectivity relations of these two processes are identical. It was proved by Smirnov [39] that critical site percolation on the triangular lattice $\mathbb{T}$ satisfies Cardy's formula, and moreover has properties of conformal invariance (see also [40, 41]). By the above observation, the dependent bond process on $\mathbb{T}$ has similar properties, and its dual process on the hexagonal lattice.

5 | Amenable Planar Graphs With One End

In this section, we prove Theorem 1.1(a) for amenable, one-ended graphs; see Remark 1.1 for an explanation of part (b) of the theorem. It is standard that such graphs are properly embeddable in the Euclidean plane, denoted $\mathcal{H}$ in this section.

Recall first that, for any infinite, quasi-transitive, amenable graph G, and invariant, insertion-tolerant measure μ, the number N_G of infinite open clusters satisfies $\mu(N_G \le 1) = 1$ (see [4, Thm. 7.9] for the transitive case, the quasi-transitive case is similar). As in Section 2.2, $\overline{N}_G$ denotes the number of infinite closed clusters.

Lemma 5.1. Let $M = (V, E) \in \mathcal{Q}$ be amenable, one-ended, and embedded canonically in $\mathcal{H}$, and let (G_1, G_2) be a matching pair derived from M. Let (Ω_V, μ) be an ergodic, insertion-tolerant site percolation on M satisfying the FKG lattice condition (6). Then

$$\mu\big((N_M, \overline{N}_M) = (1,1)\big) = \mu\big((N_{G_1}, \overline{N}_{G_2}) = (1,1)\big) = 0. \tag{10}$$

A pair γ, γ' of isometries of $\mathbb{R}^2$ is said to act in a *doubly periodic manner* on G (in its canonical embedding) if they generate a subgroup of Aut(G) that is isomorphic to $\mathbb{Z}^2$, and the embedding is called *doubly periodic* if such a pair exists. In preparation for the proof of Lemma 5.1, we note the following.

Theorem 5.1. Let $G \in \mathcal{Q}$ be amenable and one-ended. A canonical embedding of G in $\mathbb{R}^2$ is doubly periodic.

Proof. This may be proved in a number of ways, including using either Bieberbach's theorem on crystalline groups [42, 43] or Selberg's lemma [44]. Instead, we use a more direct route via the main theorem of Seifter and Trofimov [26] (see Theorem 3.1(d)).

Viewed as a graph, G has quadratic growth. This standard fact holds as follows. By [18, Thm. 1.1], G has either linear, or quadratic, or exponential growth. As noted at [45, Thm. 9.3(b)], being one-ended, it cannot have linear growth. Finally, we rule out exponential growth. Since G is quasi-transitive, there exists $R < \infty$ such that, for all edges $\langle x, y\rangle$ of G, the distance between x and y in $\mathbb{R}^2$ is no greater than R. Therefore, the n-ball centred at vertex v is contained in $B_n(v) := v + [-nR, nR]^2$. By quasi-transitivity again, there exists $A < \infty$ such that, for all v, $B_n(v)$ contains no more than $A(nR)^2$ vertices.

The theorem of [26] may now be applied to find that $\mathrm{Aut}(G)$ has a finite-index subgroup F isomorphic to $\mathbb{Z}^2$. Thus F is generated by a pair of automorphisms which, by Theorem 3.1(c), extend to isometries of the embedding of G. □

Proof of Lemma 5.1. By insertion-tolerance and ergodicity, the four random variables featuring in (10) are each μ-a.s. constant and take values in $\{0, 1\}$. By Theorem 5.1 and [46, Thm. 1.5],

$$\mu\big((N_{G_1}, \overline{N}_{G_2}) = (1,1)\big) = 0. \tag{11}$$

Arguments related to but weaker than [46, Thm. 1.5] are found in [3, 47–50]. Note that [46, Thm. 1.5] deals with bond percolation on planar graphs, whereas (11) is concerned with site percolation on non-planar graphs. The site model may be handled either by adapting the arguments of [46] to site models, or by applying [46, Thm. 1.5] to the one-dependent bond model constructed in the manner described in Section 4 (see (8) and Proposition 4.2). Non-planarity is avoided by working with the facial graphs of Section 4. The remaining part of (10) follows from the fact that $\overline{N}_{M_*} = 1$ μ-a.s. on the event $\{\overline{N}_M = 1\}$. □

Corollary 5.1. Let $G \in \mathcal{Q}$ be amenable and one-ended, and consider site percolation on G. Then $\mathbb{P}_{\frac12}(N = 0) = 1$.

Proof. Suppose that $\mathbb{P}_{\frac12}(N \geq 1) > 0$, so that $\mathbb{P}_{\frac12}(N \geq 1) = 1$ by ergodicity. By amenability and symmetry, we have that $\mathbb{P}_{\frac12}(N = \overline{N} = 1) = 1$. This contradicts Lemma 5.1. □

Lemma 5.2. Let $M = (V, E) \in \mathcal{Q}$ be amenable, one-ended, and embedded canonically in $\mathcal{H}$, and let (G_1, G_2) be a matching pair derived from M. We have for site percolation that $\mathbb{P}_p(\overline{N}_{G_2} = 1) = 1$ for $p < p_c^{\mathrm{site}}(G_1)$.

Proof. Let $p \in (0, p_c^{\mathrm{site}}(G_1))$ be such that $\mathbb{P}_p(\overline{N}_{G_2} = 1) < 1$. By amenability and ergodicity, we have that

$$\mathbb{P}_p(\overline{N}_{G_2} = 0) = 1. \tag{12}$$

Therefore, $\mathbb{P}_p(N_{G_1} = \overline{N}_{G_2} = 0) = 1$. There is a standard geometrical argument based on subcritical exponential decay that leads to a contradiction, as follows.

Fix a vertex v_0 of $M = (V, E)$, and let γ be a semi-infinite geodesic of M with endvertex v_0. Let $n \geq 1$, and let $\Lambda_n = \{u \in V : d_M(u, v_0) \leq n\}$. By [9, Prop. 2.1], if $\partial\Lambda_n$ intersects no infinite closed path of G_2, there exists some open circuit of G_1 with Λ_n in its inside. There exists $c = c(M) > 0$ such that, if the last event occurs, then for some $k \geq 1$ and some $v \in \gamma \cap \partial\Lambda_{n+k}$, we have that v lies in an open path of G_1 of length at least $c(n+k)$. By [51, Thm. 3] for example, and (12), there exist $A, a > 0$ such that,

$$1 \leq \sum_{k\geq 1} A e^{-a(n+k)}.$$

This cannot hold for large n, and the lemma is proved. □

We turn to Equation (1). In this amenable case, this is equivalent to the following extension of classical results of Sykes and Essam [8] (see also [10]).

Theorem 5.2. Let $M = (V, E) \in \mathcal{Q}$ be amenable, one-ended, and embedded canonically in $\mathcal{H}$, and let (G_1, G_2) be a matching pair derived from M. Then

$$p_c^{\mathrm{site}}(G_1) + p_c^{\mathrm{site}}(G_2) = 1.$$

Proof. By Lemma 5.1, whenever $p > p_c^{\mathrm{site}}(G_1)$, we have $1 - p \leq p_c^{\mathrm{site}}(G_2)$, which implies $p_c^{\mathrm{site}}(G_1) + p_c^{\mathrm{site}}(G_2) \geq 1$. By Lemma 5.2, whenever $p < p_c^{\mathrm{site}}(G_1)$, we have $1 - p \geq p_c^{\mathrm{site}}(G_2)$, which implies $p_c^{\mathrm{site}}(G_1) + p_c^{\mathrm{site}}(G_2) \leq 1$. □

6 | Non-Amenable Graphs With One End

In this section, we prove Theorem 1.1(a) for non-amenable, one-ended graphs $G = (V, E) \in \mathcal{Q}$; see Remark 1.1 for an explanation of part (b) of the theorem. Recall from Section 2.2 that N (respectively, $\overline{N}$) denotes the number of 1-clusters (respectively, 0-clusters).

Lemma 6.1. Let $M = (V, E) \in \mathcal{Q}$ be one-ended and embedded canonically in the hyperbolic plane, and let (G_1, G_2) be a matching pair derived from M. For $\omega \in \Omega_V$,

$$\mathbb{P}_p\big((N_{G_1}, \overline{N}_{G_2}) \in \{(0,1), (1,0), (\infty,\infty)\}\big) = 1.$$

Proof. We fix a canonical embedding of M. By Proposition 4.2,

$$N_{G_1}(\omega) = N_{\widehat{G}_1}(\beta_{\omega_1}), \qquad N_{G_2}(1-\omega) = N_{\widehat{G}_1^+}(\beta_{\omega_1}^+).$$

By Lemma 4.1, the law of β_{ω_1} is weakly deletion-tolerant, weakly-insertion tolerant, and ergodic, and the claim follows by Lemma 3.3. □

Proof of Theorem 1.1(a). By Lemmas 3.5 and 3.7, we have the following for site percolation on either G_1 or G_2:

$$\begin{aligned} \text{if } p < p_c, &\quad \mathbb{P}_p(N = 0) = 0 \\ \text{if } p_c < p < p_u, &\quad \mathbb{P}_p(N = \infty) = 1 \\ \text{if } p > p_u, &\quad \mathbb{P}_p(N = 1) = 1 \end{aligned}$$

where p_c, p_u are the critical values appropriate to the graph in question.

By Lemma 6.1, $N_{G_1} = 1$ if and only if $\overline{N}_{G_2} = 0$, whence $p_u(G_1) = 1 - p_c(G_2)$. □

Corollary 6.1. Let $G \in \mathcal{Q}$ be one-ended and embedded canonically in $\mathcal{H}$, and suppose G is non-amenable. Then

$$\mathbb{P}_p\big((N, \overline{N}) \in \{(0,0),(0,1),(1,0),(0,\infty),(\infty,0),(\infty,\infty)\}\big) = 1.$$

Proof. By Lemma 3.2, $\mathbb{P}_p$-a.s. the pair $(N, \overline{N})$ takes some given value in the set $\{0,1,\infty\}^2$. We need to eliminate the vectors $(1,1)$, $(1,\infty)$, and $(\infty,1)$. If either of the vectors $(1,1)$ and $(1,\infty)$ have strictly positive probability, then $\mathbb{P}_p(N = 1, \overline{N}_* \geq 1) > 0$, in contradiction of Lemma 6.1 applied to the matching pair (G, G_*). By symmetry, $\mathbb{P}_p((N, \overline{N}) \neq (\infty, 1)) = 1$, and the corollary follows. □

7 | Proof of Theorem 1.2

Let G be a graph satisfying the assumptions of the theorem. We work with the largest finite connected subgraph G_B of G contained in a large bounded region B (with boundary ∂B) of the natural geometry of G, and shall let B expand to fill the space. The numbers of finite faces, vertices, edges of G_B satisfy Euler's formula: $f_B + v_B = e_B + 1$. Since the smallest possible face is a triangle, we have $f_B \leq \frac{2}{3}e_B$; since the degree of interior vertices is 7 or more, there exists $c > 0$ such that $e_B \geq \frac{7}{2}(v_B - c|\partial B|)$. This contradicts Euler's formula unless $e_B/|\partial B|$ is bounded above, which is to say that the natural geometry is the hyperbolic plane. Hence, G is non-amenable. By [15, Thm. 2], we have $p_c^{\text{site}} = p_c^{\text{site}}(G) < \frac{1}{2}$.

By the symmetry of the interval $(p_c^{\text{site}}, 1 - p_c^{\text{site}})$ around $\frac{1}{2}$, it suffices to show that $\mathbb{P}_p(N = \infty) = 1$ for $p \in (p_c^{\text{site}}, 1 - p_c^{\text{site}})$. This in turn is implied by Lemma 3.1 and the inequality

$$1 - p_c^{\text{site}} \leq p_u^{\text{site}}. \tag{13}$$

Inequality (13) holds by (2) when is G non-amenable and one-ended. In the remaining case when G has infinitely many ends, (13) is trivial since $p_u^{\text{site}} = 1$ by Theorem 3.3.

8 | Proof of Theorem 1.3

Let G be a graph satisfying the assumptions of the theorem, and embedded canonically. By Lemma 3.2, symmetry, and the assumption $\mathbb{P}_{\frac{1}{2}}(N \geq 1) = 1$,

$$\mathbb{P}_{\frac{1}{2}}\Big((N, \overline{N}) \in \{(1,1),(\infty,\infty)\}\Big) = 1 \tag{14}$$

By Theorem 1.4, the following four cases may occur:

(a) G is amenable and one-ended. By Lemma 5.1, $\mathbb{P}_{\frac{1}{2}}(N = 0) = 1$. Hence, in this case, the hypothesis of the theorem is invalid.

(b) G is non-amenable and one-ended. By Corollary 6.1 and (14), subject to the percolation assumption, we have $\mathbb{P}_{\frac{1}{2}}(N = \overline{N} = \infty) = 1$.

(c) G has two ends. By Theorem 3.2, $p_c^{\text{site}} = 1$. Hence $\mathbb{P}_{\frac{1}{2}}(N = 0) = 1$, and the hypothesis is invalid.

(d) G has infinitely many ends. By Theorem 3.2, $p_u^{\text{site}} = 1$. Under the hypothesis of the theorem, it follows by symmetry that $\mathbb{P}_{\frac{1}{2}}((N, \overline{N}) = (\infty, \infty)) = 1$.

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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