Entanglement in Percolation

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Abstract

We study the existence of finite and infinite entangled graphs in the bond percolation process in three dimensions with density p. After a discussion of the relevant definitions, the entanglement critical probabilities are defined. The size of the maximal entangled graph at the origin is studied for small p, and it is shown that this graph has radius whose tail decays at least as fast as $\exp(-\alpha n/\log n)$; indeed, the logarithm may be replaced by any iterate of logarithm for an appropriate positive constant α. We explore the question of almost sure uniqueness of the infinite maximal open entangled graph when p is large, and we establish two relevant theorems. We make several conjectures concerning the properties of entangled graphs in percolation.

1 Introduction

Under what conditions does a set of arcs in $\mathbb{R}^3$ have a large entangled subset? Such an informal question arises frequently in polymer science, but its mathematical formulation poses some interesting challenges. One of these is to present an acceptable definition of a finite or infinite entangled set of arcs. Another is to establish conditions under which such entanglement occurs. In the work reported in this paper, we discuss the definition of entanglement, and we study the existence of entangled graphs in the bond percolation model.

We shall work throughout this paper with the three-dimensional cubic lattice, denoted $\mathbb{L}$. As in the theory of knots (see [14]), the entanglement of arcs is intrinsically a three-dimensional affair. Our choice of the cubic lattice is largely a matter of convenience, and other choices are possible.

Key words: entanglement percolation, percolation, entanglement, topological entanglement, boundary condition

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Let F be a finite set of edges of $\mathbb{L}$. Imagining each edge in F as an elastic but unbreakable connection, we may think of F as being 'entangled' if no part of F may be physically separated from the rest. A formal definition of a finite entangled graph is given in Section 2.

Given an infinite set G of edges of $\mathbb{L}$, a basic question for certain applications is whether or not G contains entangled graphs on all scales. This leads us to the notion of an 'infinite entangled graph'. There are various possible ways of defining such an object, and we discuss this in Section 2. We have been struck by the apparent absence in the topology literature of a study of topological entanglement. For references to entanglement in the physics literature, the reader is referred to the bibliographies in [6] and [12].

In many physical systems, one is concerned with a family of arcs in $\mathbb{R}^3$ which have been sampled from a given probability measure. One of the simplest such measures is product measure. In the case of $\mathbb{L}$, this means that each edge of $\mathbb{L}$ is retained (or declared 'open') with a certain probability p, with different edges being retained independently of one another. The resulting random set of open edges has been studied in depth under the title 'bond percolation' (see [8]), and its theory has a multifaceted relevance to the study of disordered physical systems.

The question of entanglement in percolation was posed first in [12], where certain numerical studies concerning a hypothetical 'entanglement critical probability' were reported. The question implicit in [12] was to determine for which values of p the bond percolation model on $\mathbb{L}$ contains entangled graphs on all scales. The principal numerical conclusion of [12] was that large entangled graphs occur for some values of p satisfying $p < p_c$, where p_c is the usual critical probability (that is, the threshold value of p for the property that there exists an infinite connected graph of open edges). The above strict inequality was made rigorous in [2], but no formal definition or discussion of infinite entanglement was presented there.

It was proved more recently in [11] that no infinite entangled graphs exist for sufficiently small positive values of p. More explicitly, it was proved that, for small p, the origin of $\mathbb{L}$ is almost surely contained in the inside of some sphere of $\mathbb{R}^3$ which intersects no open edge of the percolation process. Here and later, the word 'sphere' is used in its topological sense.

Our general target here is to present and discuss possible definitions for finite and infinite entangled graphs in $\mathbb{L}$, and to study the existence (or not) of such objects in bond percolation on $\mathbb{L}$. Such definitions appear in Section 2. In contrast with the case of *finite* entangled graphs, some judgement is required in order to achieve a 'correct' definition of an infinite entangled graph, and it seems likely that an appropriate definition will depend on the particular physical application. We shall present two 'extremal definitions', and we discuss how these definitions are related to the concepts of 'free' and 'wired' boundary conditions borrowed from the theory of statistical physics.

Having introduced a theory of entanglement, we apply this in the context of percolation by concentrating on three questions. Firstly, what can be said about the existence and uniqueness of an 'entanglement critical probability'? We have been unable to prove that all 'reasonable' definitions of infinite entanglement lead to the same critical probability.

Secondly, when p is smaller than the entanglement critical probability, how large is the maximal entangled graph at the origin? Motivated by results concerning connected clusters in percolation (see [8], Chapter 5), it seems reasonable to expect the tail of the distribution of the radius of the finite entangled graph E at the origin to decay exponentially. Progress in this direction is presented in Theorem 3.2, where it is proved that, for p sufficiently small, the tail decays 'near-exponentially'. Specifically, for all $k \geq 1$ and for p sufficiently small, we have that

$$P_p\Big(\mathrm{rad}(E) > r\Big) \leq \exp\left(-\frac{\alpha_k r}{\lambda_k(r)}\right),$$

where $\mathrm{rad}(E)$ is the radius of E, λ_k is the kth iterate of logarithm, and $\alpha_k = \alpha_k(p) > 0$ for small p. This is achieved via an inequality introduced in [7] to obtain comparable results for random-cluster models.

Thirdly, we ask whether, for large p, there exists almost surely a *unique* infinite maximal entangled graph. Whereas the techniques of [4] have answered many such questions for *connected* graphs, the situation for entangled graphs (as for infinite 'rigid' graphs; see [10]), is much less clear. Theorems 3.3 and 3.4 represent partial progress towards answering this question. The first theorem states that, if p is sufficiently close to 1, there exists almost surely a unique infinite maximal entangled graph; this result is valid for all reasonable choices of the definition of infinite entanglement, and furthermore the infinite maximal entangled graph is identical for all such choices. The second theorem asserts the uniqueness of the infinite maximal entangled graph when p is greater than the connectivity critical probability p_c, for a *particular* definition of infinite entanglement.

Let e_n be the number of entangled graphs of n lattice edges containing the origin. It is clear that e_n is at least as large as the number of connected graphs of size n containing the origin ('animals'), whence $e_n \geq \mu^n$ for some constant $\mu > 1$. We prove in Theorem 3.1 that

$$e_n \leq \exp\left(An + \tfrac{3}{8} n \log n\right)$$

for some constant A; we pose the question of determining the true rate of growth of e_n.

2 Definitions and Preliminary Results

We let $\mathbb{Z}^3$ be the set of all 3-vectors $x = (x_1, x_2, x_3)$ of integers, and we define the *cubic lattice* to be the set of pairs

$$\mathbb{L} = \Big\{\{x, y\} \subseteq \mathbb{Z}^3 : \|x - y\| = 1\Big\},$$

where $\|\cdot\|$ denotes Euclidean distance. The elements of $\mathbb{Z}^3$ and $\mathbb{L}$ are called *vertices* and *edges*, respectively. Thus we define the lattice by its set of edges, a definition whose convenience when working with entanglement will become evident. A *graph* is a non-empty set of edges $G \subseteq \mathbb{L}$, and a *subgraph* of a graph G is a non-empty subset of G. The *vertex set* of a graph G is the set $V(G) = \bigcup_{e \in G} e$. We say that a graph G *contains* the vertex x if $x \in V(G)$. The *origin* of $\mathbb{L}$ is the vertex $O = (0,0,0)$. Two edges are called *adjacent* if they have exactly one common vertex.

We shall consider bond percolation on the lattice $\mathbb{L}$. The appropriate sample space is the set $\Omega = \{0,1\}^{\mathbb{L}}$, which we equip with the product σ-field. For $p \in [0,1]$, we let $P_p^{\mathbb{L}}$ be the product measure on Ω with parameter p. For $e \in \mathbb{L}$ and $\omega \in \Omega$, we call e *open* (in ω) if $\omega(e) = 1$, and we call e *closed* otherwise. Similarly, if $E \subseteq \mathbb{L}$, we call E *open* if every element of E is open.

We write $K(\omega)$ for the set of open edges of the configuration $\omega \in \Omega$. In percolation theory, one is usually concerned with the connected components of the graph K. A quantity of particular interest is the *percolation probability*

$$\theta(p) = P_p^{\mathbb{L}}(K \text{ has an infinite connected subgraph containing } O),$$

which gives rise to the *critical probability*

$$p_c = \sup\{p : \theta(p) = 0\}.$$

See [8] for more details of percolation theory.

Our aim in the present paper is to study 'entangled' subgraphs of K in a way analogous to the existing theory of connected subgraphs. The first step in this direction is to formulate a precise definition of an entangled graph; this will be facilitated by some further notation. For a finite subset $A = \{a_1, a_2, \dots, a_k\}$ of $\mathbb{R}^3$, we define its *(closed) convex hull* $\langle A \rangle$ by

$$\langle A \rangle = \left\{ \sum_i \lambda_i a_i : \lambda_i \geq 0 \text{ for each } i, \text{ and } \sum_i \lambda_i = 1 \right\}.$$

Note that, for an edge $e \in \mathbb{L}$, $\langle e \rangle$ is the closed unit line segment joining the two vertices of e. For a graph E we write $[E] = \bigcup_{e \in E} \langle e \rangle$. (This definition applies to *graphs* only; the same notation will be used, with a slightly different meaning, for sets for plaquettes; see Section 5.) For any closed set R of $\mathbb{R}^3$ and any $\epsilon > 0$ we define the *(open) ϵ-neighbourhood* of R, denoted $R^{\{\epsilon\}}$, to be the set of points at Euclidean distance strictly less than ϵ from R; that is to say,

$$R^{\{\epsilon\}} = \{x \in \mathbb{R}^3 : \|x - r\| < \epsilon \text{ for some } r \in R\}.$$

By a *d-ball* (respectively *d-sphere*) we mean a closed simplicial complex in $\mathbb{R}^3$ which is homeomorphic to $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$ (respectively $\{x \in \mathbb{R}^{d+1} : \|x\| =$

$1\}$). (A simplicial complex is a compact subset of $\mathbb{R}^3$ with certain regularity properties; see [16] for a formal definition.) A 0-ball we refer to as a *point*, a 1-ball as an *arc*, a 2-ball as a *disc*, a 3-ball as a *ball*, a 1-sphere as a *loop*, and a 2-sphere as a *sphere*. Let R be a d-ball and let ϕ be a homeomorphism from $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$ to R. We define the *boundary* of R, written ΔR, to be the $(d-1)$-sphere $\phi(\{x \in \mathbb{R}^d : \|x\| = 1\})$. The *interior* of R is the set $R \setminus \Delta R$.

If S is a sphere, we define its *inside* to be the bounded connected component of $\mathbb{R}^3 \setminus S$, and its *outside* to be the unbounded connected component of $\mathbb{R}^3 \setminus S$. We use the term *plane* to denote a piecewise-linear embedding of $\mathbb{R}^2$ in $\mathbb{R}^3$ which is locally flat at infinity (see [3] for a definition of this term) when regarded as an embedding of a 2-sphere in a 3-sphere; if P is a plane, the two connected components of $\mathbb{R}^3 \setminus P$ are called its *half-spaces*. We define the *radius* of a set $R \subseteq \mathbb{R}^3$ by

$$\mathrm{rad}(R) = \sup\{\|x\| : x \in R\}.$$

(We shall apply this definition only to a set R which contains O, or to a sphere having O in its inside.)

If R is a subset of $\mathbb{R}^3$ and S is a sphere, we say that S *separates* R if $R \cap S = \emptyset$ and in addition R intersects both the inside and the outside of S. Similarly, if P is a plane, we say that P *separates* R if $R \cap P = \emptyset$ and R intersects both half-spaces of P.

We turn now towards a formal definition of the concept of entanglement. Our approach will be based on the observation made in [11] that any graph E which we might wish to regard as being entangled has the property that there exists no sphere which separates it. We begin with an observation concerning *finite* graphs.

Proposition 2.1 *Let F be a finite graph. The following three statements are equivalent.*

(i) $[F]$ *is separated by no sphere.*

(ii) $[F]$ *is separated by no plane.*

(iii) *There exists no piecewise-linear homeomorphism of $\mathbb{R}^3$ to itself such that the image of $[F]$ does not intersect $\mathbb{R}^2 \times \{0\}$ but intersects both $\mathbb{R}^2 \times (-\infty, 0)$ and $\mathbb{R}^2 \times (0, \infty)$.*

The statements of this proposition are also equivalent to a certain fourth statement which uses the jargon of knot theory. We do not explore this in detail here, but summarise it as follows. One may represent a finite graph by means of a planar 'diagram', and one may define a system of moves analogous to Reidermeister moves for knot diagrams (see [14]). It turns out that the assertions of Proposition 2.1 are equivalent to the following statement:

> A diagram of $[F]$ may not be transformed into a disconnected diagram via a sequence of moves.

Sketch Proof of Proposition 2.1. We present informal accounts of the required arguments; a more formal account would be long but uninformative. Writing $\overline{\text{(i)}}$, $\overline{\text{(ii)}}$, $\overline{\text{(iii)}}$ for the negations of the three assertions, we shall show that $\overline{\text{(i)}}$ implies $\overline{\text{(ii)}}$, $\overline{\text{(ii)}}$ implies $\overline{\text{(iii)}}$, and $\overline{\text{(iii)}}$ implies $\overline{\text{(i)}}$. Firstly, note that any sphere separating $[F]$ may be turned into a plane by attaching a long thin infinite tube which does not intersect $[F]$. There are standard topological arguments which may be used to show that $\overline{\text{(ii)}}$ implies $\overline{\text{(iii)}}$; see [3]. Finally, suppose there exists a homeomorphism ϕ as described in (iii), and let $H(r)$ denote the sphere which is the boundary of the ball $[-r,r]^2 \times [0,r]$. Since F is finite, we may find r sufficiently large that $H(r)$ separates $\phi(F)$. We then apply the inverse of ϕ to find that (i) does not hold. □

We write $\mathcal{F}$ for the set of all finite graphs satisfying the statements of Proposition 2.1. We think of $\mathcal{F}$ as the set of all finite 'entangled' graphs, although we prefer not to give a formal definition of this term, in order to avoid possible confusion with certain forthcoming definitions for infinite graphs. Note that $\mathcal{F}$ contains every finite connected graph.

We turn now to infinite graphs. Examples were given in [11] of infinite graphs which one would wish to regard as being 'entangled', together with other graphs for which the use of the term is questionable. It turns out that there is more than one plausible definition of entanglement for infinite graphs, and the most appropriate choice may depend on the particular application. Furthermore, challenging topological questions arise in the general study of entanglements of infinite structures. Rather than pursuing such questions directly, we adopt the following approach. We shall describe a class of families of graphs (called 'entanglement systems'), and we claim that any reasonable definition of 'entanglement' leads to a family of graphs belonging to this class. We shall also characterise two particular families within this class which are extremal in the sense that any entanglement system contains the smaller and is contained in the larger.

Let $\mathcal{E}$ be a family of (finite or infinite) graphs in $\mathbb{L}$. We think of $\mathcal{E}$ as a candidate for the set of all 'entangled graphs', according to some definition. Members of $\mathcal{E}$ are called $\mathcal{E}$-graphs; if $A \subseteq \mathbb{L}$, and B is a subgraph of A lying in $\mathcal{E}$, we call B a *$\mathcal{E}$-subgraph* of A. We call $\mathcal{E}$ an *entanglement system* if the following conditions hold:

(E1) The intersection of $\mathcal{E}$ with the set of finite graphs in $\mathbb{L}$ is exactly $\mathcal{F}$.

(E2) For any $A_1, A_2, \ldots \in \mathcal{E}$ with $V(A_i) \cap V(A_j) \neq \emptyset$ for $i \neq j$, we have that $\bigcup_i A_i \in \mathcal{E}$.

(E3) If $A \in \mathcal{E}$ then $[A]$ is separated by no sphere.

(Note that our definition of an entanglement system differs from the definition of an 'e-system' given in [11].) We call an entanglement system *measurable* if in addition:

(E4) For all $x \in \mathbb{Z}^3$, the set of configurations $I_x(\mathcal{E}) = \{\omega \in \Omega : K(\omega)$ has an infinite $\mathcal{E}$-subgraph containing $x\}$ is measurable.

We make some remarks concerning these definitions. Firstly, we claim that properties (E1)–(E3) are reasonable requirements for the property of 'entanglement'. Secondly, it might seem reasonable to replace the word 'sphere' in (E3) by 'plane'; it may be seen that this would lead to a restriction of the definition, and the forthcoming results would then be weakened. Thirdly, we remark that condition (E4) above will be necessary for the study of entanglement in percolation; one may construct entanglement systems $\mathcal{E}$ which are not measurable.

Fourthly, the following proposition states that entanglement systems are closed under the operation of taking uncountable unions of graphs with pairwise intersecting vertex sets.

Proposition 2.2 *Let $\mathcal{E}$ be an entanglement system, and let $\{A_i : i \in I\}$ be a subset of $\mathcal{E}$ satisfying $V(A_i) \cap V(A_j) \neq \emptyset$ for $i \neq j$. The union $\bigcup_{i \in I} A_i$ belongs to $\mathcal{E}$.*

Proof. Let $\{A_i : i \in I\}$ satisfy the conditions of the proposition, and write $A = \bigcup_{i \in I} A_i$. For $x \in V(A)$, let D_x be the union of all those A_i satisfying $x \in V(A_i)$, say $D_x = \bigcup_{i \in I_x} A_i$. For each $e \in D_x$, we may find $i(x, e) \in I_x$ such that $e \in A_{i(x,e)}$. It follows that

$$D_x = \bigcup_{e \in D_x} A_{i(x,e)},$$

a countable union, whence $D_x \in \mathcal{E}$ by condition (E2). Also

$$A = \bigcup_{x \in \mathbb{Z}^3} D_x,$$

a countable union which belongs to $\mathcal{E}$ by (E2). □

Finally, we comment that (E2) may appear unnecessarily strong. A possible alternative is the following. If $\mathcal{A}$ is a collection of subgraphs of $\mathbb{L}$, we define the 'intersection graph' $\mathcal{G}(\mathcal{A})$ having vertex set $\mathcal{A}$, and an edge $\{A, B\}$ whenever $V(A) \cap V(B) \neq \emptyset$ and $A \neq B$.

(E2′) For any $A_1, A_2, \ldots \in \mathcal{E}$ such that the graph $\mathcal{G}(\{A_1, A_2, \ldots\})$ is connected, we have that $\bigcup_i A_i \in \mathcal{E}$.

It is elementary to show that (E2) and (E2′) are equivalent. Furthermore, one may prove a corresponding version of Proposition 2.2, namely that $\bigcup_{i \in I} A_i \in \mathcal{E}$ whenever the A_i are such that $A_i \in \mathcal{E}$ for all i and the graph $\mathcal{G}(\{A_i : i \in I\})$ is connected.

Let $\mathcal{E}$ be an entanglement system, and let $x \in \mathbb{Z}^3$ and $\omega \in \Omega$. We define $C_x(\mathcal{E})$ to be the union of all open graphs lying in $\mathcal{E}$ and containing the vertex x. By Proposition 2.2, provided $C_x(\mathcal{E})$ is non-empty, it is itself a member of $\mathcal{E}$. We abbreviate $C_O(\mathcal{E})$ to $C(\mathcal{E})$. We define an $\mathcal{E}$*-component* to be a maximal $\mathcal{E}$-subgraph of $K(\omega)$. It is readily verified that the set of graphs $\{C_x(\mathcal{E}) : x \in \mathbb{Z}^3\} \setminus \{\emptyset\}$ is precisely the set of $\mathcal{E}$-components, and that these graphs partition K.

The following two families of graphs will be central to our study of entanglement systems:

$$\begin{aligned}\mathcal{E}_0 = \{A \subseteq \mathbb{L} : \; & A \neq \emptyset \text{ and every finite subgraph of } A \text{ is a} \\ & \text{subgraph of some } \mathcal{F}\text{-subgraph of } A\};\\ \mathcal{E}_1 = \{A \subseteq \mathbb{L} : \; & A \neq \emptyset \text{ and } [A] \text{ is separated by no sphere}\}.\end{aligned}$$

The following proposition states that $\mathcal{E}_0$ and $\mathcal{E}_1$ are minimal and maximal entanglement systems respectively.

Proposition 2.3 *The sets $\mathcal{E}_0$ and $\mathcal{E}_1$ are measurable entanglement systems, and every entanglement system $\mathcal{E}$ satisfies $\mathcal{E}_0 \subseteq \mathcal{E} \subseteq \mathcal{E}_1$.*

In our proof of Proposition 2.3 we shall make use of the following fact, which has other applications also.

Proposition 2.4 *For all $\omega \in \Omega$, the set $C(\mathcal{E}_1)$ is finite if and only if there exists a sphere lying in $\mathbb{R}^3 \setminus [K(\omega)]$ with O in its inside.*

We defer the proofs of Propositions 2.3 and 2.4 to the end of this section. We shall now make some remarks about $\mathcal{E}_0$ and $\mathcal{E}_1$. The definition of $\mathcal{E}_0$ appears fairly natural, and is analogous to the definition of infinite rigid graphs used in [10]. Note however that there exist graphs which do not belong to $\mathcal{E}_0$ but which might be regarded as entangled. See, for example, the graph illustrated in Figure 1. On the other hand, $\mathcal{E}_1$ contains graphs which we might prefer not to regard as entangled, such as any graph which is the union of two disjoint doubly infinite paths.

Note that, for any given $\omega \in \Omega$, there is at most one infinite $\mathcal{E}_1$-component. This is an immediate consequence of the definition of $\mathcal{E}_1$, on noting that the union of all infinite $\mathcal{E}_1$-components is separated by no sphere.

For any measurable entanglement system $\mathcal{E}$, we define

$$\eta^{\mathcal{E}}(p) = P_p^{\mathbb{L}}\Big(|C(\mathcal{E})| = \infty\Big),$$

and the corresponding critical probability

$$p_{\mathrm{e}}^{\mathcal{E}} = \sup\{p : \eta^{\mathcal{E}}(p) = 0\}.$$

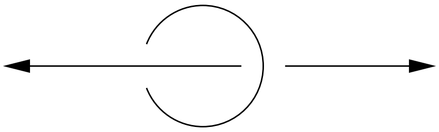

Figure 1: This graph comprising a loop around a doubly-infinite path does not belong to $\mathcal{E}_0$.

We use the abbreviations $\eta^j = \eta^{\mathcal{E}_j}$ and $p_{\mathrm{e}}^j = p_{\mathrm{e}}^{\mathcal{E}_j}$ for $j = 0, 1$. We have by Proposition 2.3 that $\eta^0 \le \eta^{\mathcal{E}} \le \eta^1$, and therefore $p_{\mathrm{e}}^1 \le p_{\mathrm{e}}^{\mathcal{E}} \le p_{\mathrm{e}}^0$, for any entanglement system $\mathcal{E}$. Since every connected graph lies in $\mathcal{E}_0$, we have also that $p_{\mathrm{e}}^0 \le p_{\mathrm{c}}$, and in particular $p_{\mathrm{e}}^0 < 1$. The inequality $p_{\mathrm{e}}^0 \le p_{\mathrm{c}}$ may be strengthened to strict inequality as described in [2]. It was proved in [11] that $0 < p_{\mathrm{e}}^1$. We summarise these results as follows:

$$0 < p_{\mathrm{e}}^1 \le p_{\mathrm{e}}^{\mathcal{E}} \le p_{\mathrm{e}}^0 < p_{\mathrm{c}} < 1 \qquad \text{for any entanglement system } \mathcal{E}.$$

It is an important unanswered question whether or not p_{e}^0 and p_{e}^1 are equal.

Finally, we note that the definitions of the entanglement systems $\mathcal{E}_0$ and $\mathcal{E}_1$ may be motivated by a discussion of 'boundary conditions'. Let $B(n)$ be the subgraph of $\mathbb{L}$ consisting of all edges both of whose vertices lie in $[-n, n]^3$, and let $\partial B(n)$ be the set of vertices $V(B(n)) \cap V(\mathbb{L} \setminus B(n))$. Given the configuration ω and the corresponding graph of open edges $K(\omega)$, we define $K_n^0 = K \cap B(n)$ and $K_n^1 = K \cup (\mathbb{L} \setminus B(n))$. Thus K_n^0 (respectively K_n^1) is obtained from K by removing (respectively adding) all edges outside $B(n)$.

It is easily verified that

$$\{|C(\mathcal{E}_0)| = \infty\} = \limsup_{n\to\infty} E_n^0,$$

where

$$E_n^0 = \Big\{K_n^0 \text{ has an } \mathcal{F}\text{-subgraph which contains } O \text{ and some vertex of } \partial B(n)\Big\}.$$

For this reason, we may think of $\mathcal{E}_0$ as the family of entangled graphs generated by what might be termed 'boundary condition 0'. A similar, and indeed somewhat stronger, correspondence exists between $\mathcal{E}_1$ and 'boundary condition 1'. Specifically, it may be deduced from Proposition 2.4 that we have

$$\{|C(\mathcal{E}_1)| = \infty\} = \lim_{n\to\infty} E_n^1, \tag{1}$$

where

$$E_n^1 = \Big\{K_n^1 \text{ has an } \mathcal{F}\text{-subgraph which contains } O \text{ and some vertex of } \partial B(n)\Big\},$$

and the limit is decreasing.

We turn next to the proofs of Propositions 2.3 and 2.4, and begin with the latter, for which we shall require two topological lemmas. The following abuse of notation will be convenient: if a and b are real numbers satisfying $a < b$, and S_a and S_b are two spheres such that S_a lies in the inside of S_b, we write $S_{[a,b]}$ for the closed region of $\mathbb{R}^3$ lying between the two spheres; that is, the complement (in $\mathbb{R}^3$) of the union of the inside of S_a and the outside of S_b. We also write $S_{[-\infty,a]}$ for the union of S_a and its inside, and $S_{[a,\infty]}$ for the union of S_a and its outside.

The first lemma states than given any sphere, we may find two further spheres lying 'just inside' and 'just outside' the first.

Lemma 2.5 *Let S_0 be any sphere and let $\epsilon > 0$. There exist spheres denoted $S_{-\epsilon}$ and S_ϵ with the following properties: $S_{-\epsilon}$ lies in the inside of S_0, and $S_{[-\epsilon,0]} \subseteq {S_0}^{\{\epsilon\}}$; S_0 lies in the inside of S_ϵ, and $S_{[0,\epsilon]} \subseteq {S_0}^{\{\epsilon\}}$.*

The second lemma concerns homeomorphisms which 'retract' one sphere onto another.

Lemma 2.6 *Let $a < b < c$, and let S_a, S_b and S_c be spheres such that S_a lies in the inside of S_b, and S_b lies in the inside of S_c.*

- (i) *There exists a piecewise-linear homeomorphism from $S_{[a,\infty]}$ to $S_{[b,\infty]}$ whose restriction to $S_{[c,\infty]}$ is the identity.*
- (ii) *There exists a piecewise-linear homeomorphism from $S_{[-\infty,c]}$ to $S_{[-\infty,b]}$ whose restriction to $S_{[-\infty,a]}$ is the identity.*

We omit the proofs of Lemmas 2.5 and 2.6, the validity of which may be demonstrated using standard techniques, as in [16], for example. We are now ready to prove Proposition 2.4.

Proof of Proposition 2.4. It is immediate from the definition of $\mathcal{E}_1$ that the existence of a sphere with the stated properties implies that $C(\mathcal{E}_1)$ is finite. We therefore turn to the proof of the converse statement.

Let $\omega \in \Omega$ and $K = K(\omega)$. We write $\mathcal{E}_1(K)$ for the set of all $\mathcal{E}_1$-subgraphs of K, and we define

$$\widetilde{\mathcal{E}_1}(K) = \Big\{G \subseteq K : G \neq \emptyset \text{ and } [G] \text{ is separated by no sphere lying in } \mathbb{R}^3 \setminus [K]\Big\}.$$

Clearly we have $\widetilde{\mathcal{E}_1}(K) \supseteq \mathcal{E}_1(K)$, but these sets are in general unequal: for example, if G consists of two non-adjacent edges of a connected subgraph of K, then G lies in $\widetilde{\mathcal{E}_1}(K)$ but not in $\mathcal{E}_1(K)$. We recall the definition of an $\mathcal{E}_1$-component, and we define an $\widetilde{\mathcal{E}_1}$-*component* analogously to be a maximal element of $\widetilde{\mathcal{E}_1}(K)$. It is straightforward to check (by the same argument as for $\mathcal{E}_1$-components) that the $\widetilde{\mathcal{E}_1}$-components partition K. The central step in the proof of Proposition 2.4 will be to prove the following claim:

$$\begin{array}{c}\text{for any configuration } \omega, \text{ the set of } \widetilde{\mathcal{E}_1}\text{-components} \\ \text{is identical to the set of } \mathcal{E}_1\text{-components.}\end{array} \tag{2}$$

For any sphere L we write $N(L) = N(L, \omega)$ for the number of open edges $e \in K(\omega)$ such that L intersects $\langle e \rangle$. Since a sphere is a bounded subset of $\mathbb{R}^3$, $N(L)$ is necessarily finite.

Since $\widetilde{\mathcal{E}_1}(K) \supseteq \mathcal{E}_1(K)$, we have that each $\widetilde{\mathcal{E}_1}$-component is a union of $\mathcal{E}_1$-components, so all that is required is to rule out the possibility that there exists an $\widetilde{\mathcal{E}_1}$-component which is not an $\mathcal{E}_1$-graph. Suppose on the contrary that G is such an $\widetilde{\mathcal{E}_1}$-component, and let L be a sphere which separates $[G]$. Let G_1 (respectively G_2) be the subgraph of G consisting of all edges in the inside (respectively outside) of L, so that G is the disjoint union of G_1 and G_2. The sphere L has the following properties.

(i) $[G_1]$ lies in the inside of L.

(ii) $[G_2]$ lies in the outside of L.

Note that properties (i) and (ii) imply that L does not intersect $[G]$. We recall that $N(L) < \infty$. If $N(L) = 0$, L is disjoint from $[K]$, in contradiction to the assumption that G is an $\widetilde{\mathcal{E}_1}$-component. Our approach will be to make successive modifications to L so as to reduce the number of open edges which it intersects, while retaining properties (i) and (ii), and to proceed inductively until the number of such edges has been reduced to zero; this will yield a contradiction as above.

Suppose $N(L) > 0$ and let H be an $\widetilde{\mathcal{E}_1}$-component such that L intersects $[H]$. We shall use Lemma 2.6 to modify L so as to remove its intersections with $[H]$. Since H and G are distinct $\widetilde{\mathcal{E}_1}$-components, there exists a sphere, S_0 say, lying in $\mathbb{R}^3 \setminus [K]$, which separates $[H] \cup [G]$, and either $[H]$ lies in the inside of S_0 and $[G]$ lies in the outside, or *vice versa*. We shall treat the two cases separately. Firstly, let $\epsilon > 0$ be sufficiently small that ${S_0}^{\{\epsilon\}} \subseteq \mathbb{R}^3 \setminus [K]$ (that this is possible follows by a compactness argument), and let $S_{-\epsilon}$ and S_ϵ be as in Lemma 2.5. Now, if $[H]$ lies in the inside of S_0, let S_{-1} be any (small) sphere lying in the inside of $S_{-\epsilon}$ and not intersecting L, and apply part (i) of Lemma 2.6 taking $a = -1$, $b = -\epsilon$ and $c = 0$. If on the other hand $[H]$ lies in the outside of S_0, let S_1 be any (large) sphere having both S_ϵ and L in its inside, and apply part (ii) of Lemma 2.6 taking $a = 0$, $b = \epsilon$ and $c = 1$. In either case, let ϕ be the homeomorphism given under the appropriate part of Lemma 2.6. Evidently $\phi(L)$ is a sphere, and we claim that it satisfies properties (i) and (ii) above (with L replaced with $\phi(L)$), and that $N(\phi(L)) < N(L)$. Properties (i) and (ii) hold because ϕ is a homeomorphism which is the identity on $[G_1]$ and $[G_2]$. The number of edges $N(L)$ is reduced for the following reason. The part of L which intersects $[H]$ is mapped by ϕ into ${S_0}^{\{\epsilon\}}$, whose intersection with $[H]$ is empty. And it is easy to see from the properties of ϕ that any edge of K which intersects $\phi(L)$ also intersects L. As explained above, we now argue by induction to obtain a contradiction, and we have proved the claim (2).

Finally, suppose that $|C(\mathcal{E}_1)| < \infty$. By (2), $C(\mathcal{E}_1)$ is also the $\widetilde{\mathcal{E}_1}$-component at the origin. Let L be any sphere having $C(\mathcal{E}_1)$ in its inside. We repeat the argument above, making successive modifications to L so as to remove its intersections with all other $\widetilde{\mathcal{E}_1}$-components, until we have a sphere which has $C(\mathcal{E}_1)$ in its inside, and intersects no edges of K. This completes the proof. $\square$

Proof of Proposition 2.3. In order to prove that $\mathcal{E}_0$ and $\mathcal{E}_1$ are entanglement systems, we must verify that they satisfy conditions (E1)–(E3). We shall give details only of the proof that $\mathcal{E}_0$ satisfies (E2), the other statements being straightforward. Suppose that $A_1, A_2, \ldots \in \mathcal{E}_0$ satisfy the conditions of (E2), and let F be any finite subgraph of $\bigcup_i A_i$. There exists some finite set of natural numbers J such that $F = \bigcup_{i\in J} B_i$, where B_i is some finite subgraph of A_i for each $i \in J$. There exist finite sets C_i satisfying $B_i \subseteq C_i \subseteq A_i$ for each $i \in J$ such the C_i have pairwise-intersecting vertex sets. Since $A_i \in \mathcal{E}_0$, we may find $\mathcal{F}$-graphs D_i satisfying $C_i \subseteq D_i \subseteq A_i$. Since the D_i have pairwise-intersecting vertex sets, any sphere which separates $\bigcup_{i\in J} D_i$ must separate some D_i, a contradiction since $D_i \in \mathcal{F}$; it follows that $\bigcup_{i\in J} D_i \in \mathcal{F}$. Since $F \subseteq \bigcup_{i\in J} D_i \subseteq \bigcup_i A_i$, we have proved as required that $\bigcup_i A_i \in \mathcal{E}_0$.

Next we prove the measurability of $\mathcal{E}_0$ and $\mathcal{E}_1$. By the translation invariance of $\mathcal{E}_0$ and $\mathcal{E}_1$, it suffices to prove the measurability of the sets $I_O(\mathcal{E}_0)$ and $I_O(\mathcal{E}_1)$, as defined in (E4). To see that $I_O(\mathcal{E}_0)$ is measurable, note that

$$I_O(\mathcal{E}_0) = \limsup_{n\to\infty} E_n,$$

$$\text{where} \qquad E_n = \Big\{\omega \in \Omega : \ O \text{ is contained in some open } \mathcal{F}\text{-graph of size } n\Big\}.$$

The set E_n is measurable, since the number of such $\mathcal{F}$-graphs is finite. (We shall obtain an explicit upper bound for the number of such graphs in Section 4; a cruder argument suffices to show that the number is finite.) Turning to $\mathcal{E}_1$, we have by Proposition 2.4 that $I_O(\mathcal{E}_1)$ occurs if and only if there exists no sphere lying in $\mathbb{R}^3 \setminus [K(\omega)]$ with O in its inside. The appropriate set of configurations may be written as $\bigcap_{r=1}^{\infty} T_r$ where T_r is the event that there exists no such sphere with radius r or less; each T_r is a cylinder event (compare (1)).

Let $\mathcal{E}$ be any entanglement system. We shall prove next that $\mathcal{E}_0 \subseteq \mathcal{E} \subseteq \mathcal{E}_1$. Let $A \in \mathcal{E}_0$. If A is finite then it is immediate from condition (E1) that $A \in \mathcal{E}$. Otherwise we write $A = \{e_1, e_2, \ldots\}$ and define $A_i = \{e_1, e_2, \ldots, e_i\}$. By the definition of $\mathcal{E}_0$, we may find $B_i \in \mathcal{F}$ satisfying $A_i \subseteq B_i \subseteq A$, and we have $A = \bigcup_i B_i$. It follows from properties (E1) and (E2) that $A \in \mathcal{E}$. Finally, it is immediate by (E3) that $\mathcal{E} \subseteq \mathcal{E}_1$ for every entanglement system $\mathcal{E}$. $\square$

3 Results

Our first main result concerns the number of $\mathcal{F}$-graphs containing the origin. We define

$$e_n = \Big|\Big\{F \in \mathcal{F} : |F| = n \text{ and } F \text{ contains } O\Big\}\Big|.$$

Since every $\mathcal{F}$-graph is connected, e_n is bounded below by the number of connected graphs of size n containing O; this number grows (asymptotically) exponentially with n (see, for example [8], Chapter 4). The following upper bound for e_n will be proved in Section 4.

Theorem 3.1 *There exists a constant A such that*

$$e_n \leq \exp\Big(An + \tfrac{3}{8} n \log n\Big) \quad \textit{for all } n.$$

We do not know the true rate of growth of e_n, or indeed whether or not e_n grows faster than all exponential functions of n.

The main result of [11] implies that $|C(\mathcal{E}_1)| < \infty$ almost surely if p is sufficiently small but positive. Our next result is to extend this to an upper bound for the tail of the distribution of the size of $C(\mathcal{E}_1)$ for small positive p.

We write λ_k for the kth iterate of the natural logarithm function, defined for convenience thus:

$$\begin{aligned} \lambda_1(x) &= \log x, \\ \lambda_{k+1}(x) &= \max\{\log(\lambda_k(x)), 1\} \quad \text{for } k \geq 1. \end{aligned}$$

Theorem 3.2 *There exists $p_0 > 0$ such that, for every $p < p_0$ and every $k = 1, 2, \ldots,$ there exists $\alpha_k = \alpha_k(p) > 0$ such that*

$$P_p^{\mathbb{L}}\Big(\mathrm{rad}(C(\mathcal{E}_1)) > r\Big) \leq \exp\left(-\frac{\alpha_k r}{\lambda_k(r)}\right) \quad \textit{for all } r \geq 0.$$

In the light of Proposition 2.3, this inequality for $\mathcal{E}_1$ implies a corresponding inequality for any entanglement system $\mathcal{E}$, with the same functions α_k. The proof of Theorem 3.2 will be given in Section 6. The idea of the proof is to refine the method used in [11] to prove $p_{\mathrm{e}}^1 > 0$, making use of some probabilistic tools taken from [7].

Our next result is that, for p close to 1, there exists a unique infinite $\mathcal{E}$-component for any measurable entanglement system $\mathcal{E}$; furthermore this component is identical for all measurable entanglement systems.

Theorem 3.3 *There exists $p_1 < 1$ such that, whenever $p \geq p_1$, there exists almost surely an infinite open graph I with the following property. For every measurable entanglement system $\mathcal{E}$, I is the* unique *infinite $\mathcal{E}$-component.*

This result implies in particular that the functions η^0 and η^1 are equal for large p, since for $p \geq p_1$ we have

$$\eta^0(p) = P_p^{\mathbb{L}}(I \text{ contains } O) = \eta^1(p).$$

The proof of Theorem 3.3 will be given in Section 7, and depends on showing that, for $p \geq p_1$, the origin is almost surely enclosed by an infinite sequence of nested surfaces with the property that every edge of $\mathbb{L}$ lying in such a surface is open.

We conjecture that, for any entanglement system $\mathcal{E}$, there exists a unique infinite $\mathcal{E}$-component whenever $\eta^{\mathcal{E}}(p) > 0$. As remarked in Section 2, this is trivially true for the entanglement system $\mathcal{E}_1$; and Theorem 3.3 establishes the conjecture for all $\mathcal{E}$ when p is large. Our final result is a partial affirmation of the conjecture for the entanglement system $\mathcal{E}_0$: for p strictly greater than the connectivity critical probability p_{c}, we have uniqueness for $\mathcal{E}_0$. The proof is given in Section 8.

Theorem 3.4 *Suppose $p > p_{\mathrm{c}}$. There exists almost surely a unique infinite $\mathcal{E}_0$-component.*

4 The Number of Finite Entangled Graphs

Proof of Theorem 3.1. Suppose F is an $\mathcal{F}$-graph of size n containing O, and let $C_1, C_2, \ldots, C_k$ be the connected components of F. Note that, provided $n \geq 8$, every C_i satisfies $|C_i| \geq 8$. To see this, observe that, given any connected graph C of size 7 or less, we may find a sphere which does not intersect $[C]$ and which does not intersect $\langle e \rangle$ for any edge $e \in \mathbb{L}$ not sharing a vertex with C (there is a unique counterexample with 8 edges, forming a square two-dimensional loop). Note further that, after re-ordering the indices $1, 2, \ldots, k$ if necessary, we have that:

(i) C_1 contains O, and

(ii) for each $i \geq 1$, $[C_{i+1}]$ intersects the convex hull of $[C_1 \cup C_2 \cup \cdots \cup C_i]$.

If (ii) were to fail for a given i, there would exist a sphere lying just outside the convex hull of $[C_1 \cup C_2 \cup \cdots \cup C_i]$ which would separate F.

Using these facts, we may bound the number of such graphs as follows: choose C_1 containing O, then choose some vertex x_1 lying in the convex hull of $[C_1]$; choose C_2 containing x_1, and choose a vertex x_2 lying in the convex hull of $[C_1 \cup C_2]$; iterate this process. The number of connected graphs of size n containing a given vertex is bounded above by μ^n for some constant μ (see for example [8], Chapter 4); and it is straightforward to verify that the convex hull of $[C_1 \cup C_2 \cup \cdots \cup C_i]$ contains at most $(2|C_1 \cup C_2 \cup \cdots \cup C_i|)^3$ vertices. Therefore, we obtain the bound

$$e_n \leq \sum_{\mathbf{c}} \mu^{c_1}[2c_1]^3 \mu^{c_2}[2(c_1+c_2)]^3 \mu^{c_3} \cdots [2(c_1+\cdots+c_{k-1})]^3 \mu^{c_k},$$

where the sum is over all possible sequences $\mathbf{c} = (c_1, c_2, \dots, c_k)$, where k is any integer satisfying $1 \le k \le n/8$, and such that $c_i \ge 8$ for all i, and $c_1 + c_2 + \cdots + c_k = n$. We bound this expression as follows:

$$\begin{aligned} e_n &\le \mu^n \sum_{\mathbf{c}} (2n)^{3k} \\ &\le \mu^n \sum_{k=1}^{\lfloor n/8 \rfloor} \binom{n+k-1}{k-1} (2n)^{3k} \\ &\le \mu^n \sum_{k=1}^{\lfloor n/8 \rfloor} \frac{(n+k)^k}{k!} (2n)^{3k} \\ &\le a^n \sum_{k=1}^{\lfloor n/8 \rfloor} \left(\frac{n^4}{k} \right)^k \end{aligned}$$

for some constant a. The binomial coefficient in the second line is an upper bound on the number of ways of choosing $c_1, c_2, \dots, c_k$ summing to n. By considering the ratios of successive terms, we see that, for n sufficiently large, the final term in the last sum is the greatest, whence

$$e_n \le a^n \frac{n}{8} \left(\frac{n^4}{n/8} \right)^{n/8} \le \exp\left(An + \tfrac{3}{8} n \log n \right)$$

for a suitable constant A. □

5 Plaquette Percolation

The dual process to three-dimensional bond percolation is a process defined on 'plaquettes'. This dual process was studied first in [1] and applied subsequently to entanglements in [11]. We indicate in this section how surfaces of plaquettes are related to entanglement.

We define the set

$$\mathbb{P} = \Big\{ \{a,b,c,d\} \subseteq \mathbb{Z}^3 : a,b,c,d \text{ are distinct and } \\ \|a-b\| = \|b-c\| = \|c-d\| = \|d-a\| = 1 \Big\},$$

and we refer to the members of $\mathbb{P}$ as *plaquettes*. If f is a plaquette, note that the convex hull $\langle f \rangle$ is a closed square subset of $\mathbb{R}^3$ having unit side-length. For a set F of plaquettes, we write $[F] = \bigcup_{f \in F} \langle f \rangle$. (Recall the contrasting definition of $[G]$ for a graph G, namely the union of the line segments of edges in G.) In the *plaquette percolation model* on $\mathbb{P}$ with parameter q we declare each plaquette to be 'retained' with probability q or 'discarded' with probability $1-q$, with different plaquettes receiving independent designations; we write Q for the random set of retained plaquettes. More precisely, we define the sample space $\Pi = \{0,1\}^{\mathbb{P}}$, equipped with the product σ-field. For $q \in [0,1]$ we define $P_q^{\mathbb{P}}$ to be the product measure on Π with parameter q. We define the random variable Q on Π by $Q(\pi) = \{f \in \mathbb{P} : \pi(f) = 1\}$.

We call an event A ($\subseteq \Pi$) *increasing* if $\pi \in A$ whenever there exists $\pi' \in \Pi$ satisfying $\pi' \leq \pi$ and $\pi' \in A$. We call A a *cylinder event* if there exists some $\pi' \in \Pi$ and some finite subset F of $\mathbb{P}$ such that $A = \{\pi \in \Pi : \pi(f) = \pi'(f) \text{ for all } f \in F\}$; in this case, we say that A is *defined on* the set F.

There is a natural bijection between Ω and Π, as follows. Let $\mathbb{L}_+$ be the 'shifted' cubic lattice $\mathbb{L} + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and e_+ the shifted edge $e + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. For any shifted edge e_+ there is a unique plaquette f such that $\langle e\rangle \cap \langle f\rangle \neq \emptyset$, and for every plaquette there is a unique shifted edge with this property. Given $e \in \mathbb{L}$, we define $f(e)$ to be the plaquette corresponding to e_+ in this way. We observe that $f(e)$ is also the unique plaquette such that $\langle e_+\rangle$ intersects the $\frac{1}{4}$-neighbourhood $\langle f(e)\rangle^{\{1/4\}}$. For $\omega \in \Omega$, we may define a corresponding $\pi \in \Pi$ by

$$\pi(f(e)) = 1 \text{ if and only if } \omega(e) = 0. \tag{3}$$

Now let $p+q=1$, and choose ω according to the product measure $P_p^{\mathbb{L}}$. It is clear that π is governed by the product measure $P_q^{\mathbb{P}}$.

Writing K_+ for the set of shifted open edges of a configuration $\omega \in \Omega$, and Q for the set of retained plaquettes constructed from the configuration π given in (3), we have by the above remarks that the sets $[K_+]$ and $[Q]^{\{1/4\}}$ are disjoint. As indicated earlier, the above correspondence between the plaquette and bond processes is loosely referred to as 'duality'.

It was proved in [11] that $p_\mathrm{e}^1 > 0$. This was achieved by showing that, for q sufficiently close to 1, there exists almost surely a sphere contained in $[Q]^{\{1/4\}}$ and enclosing a predetermined point. It is natural to ask whether or not the corresponding statement holds with $[Q]^{\{1/4\}}$ replaced by $[Q]$. More specifically, is it the case that, for q sufficiently close to 1, $[Q]$ contains almost surely a sphere enclosing the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$? We discuss this question further below.

Let R be a random subset of $\mathbb{R}^3$ defined on the sample space Π, and define the event

$$\mathcal{S}(R) = \{R \text{ contains a sphere with } (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \text{ in its inside}\};$$

we shall be concerned only with the cases where R is one of $[Q]$, $[Q]^{\{\epsilon\}}$, $\mathbb{R}^3 \setminus K_+$. By a convenient abuse of notation, given a random set $R = R(\pi)$, we define the critical point

$$q_\mathrm{s}(R) = \sup\{q : P_q^{\mathbb{P}}(\mathcal{S}(R)) < 1\}.$$

Proposition 5.1 *We have that*

$$1 - p_\mathrm{e}^1 = q_\mathrm{s}(\mathbb{R}^3 \setminus [K_+]) = q_\mathrm{s}([Q]^{\{\epsilon\}}) < 1 \quad \textit{for all } 0 < \epsilon < \tfrac{1}{2}.$$

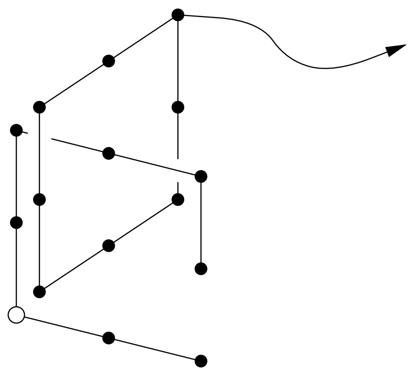

Figure 2: The marked vertex $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is enclosed by a sphere lying in the complement of the graph, but not by a sphere of retained dual plaquettes.

PROOF. We claim first that

$$\mathcal{S}(\mathbb{R}^3 \setminus [K_+]) = \mathcal{S}([Q]^{\{\epsilon\}}) \quad \text{for all } 0 < \epsilon < \tfrac{1}{2},$$

and this may be proved as follows. Given a sphere in $\mathbb{R}^3 \setminus [K_+]$, we may perturb it slightly so that it avoids all vertices of $\mathbb{L}_+$, and then we may find a homeomorphism of $\mathbb{R}^3$ to itself which maps the resulting sphere into $[Q]^{\{\epsilon\}}$. The second equality of the proposition follows from this observation.

Secondly, Proposition 2.4 implies that

$$\mathcal{S}(\mathbb{R}^3 \setminus [K_+]) = \Big\{K_+ \text{ has no infinite } \mathcal{E}_1\text{-subgraph containing } (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\Big\},$$

and the first equality of the proposition follows. The inequality

$$q_{\mathrm{s}}([Q]^{\{1/4\}}) < 1$$

was proved in [11]. □

Turning to the event $\mathcal{S}([Q])$, we note that $\mathcal{S}([Q]) \subseteq \mathcal{S}([Q]^{\{\epsilon\}})$ for $\epsilon > 0$. Perhaps surprisingly, these events are not equal, as may be seen by inspecting the graph illustrated in Figure 2. In this case, there exists a sphere enclosing $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ which is contained in $\mathbb{R}^3 \setminus [K_+]$, but there exists no such sphere lying in $[Q]$. We are led to pose the question of deciding whether or not the strict inequalities

$$q_{\mathrm{s}}([Q]^{\{\epsilon\}}) < q_{\mathrm{s}}([Q]) < 1$$

are valid for $0 < \epsilon < \frac{1}{2}$.

The existence of graphs such as that in Figure 2 may have been overlooked in [1], where it is suggested that the existence of a 'disc of plaquettes' separating the top and bottom faces of a cuboid block is equivalent to the absence of an 'entangled connection' between the top and bottom faces. A counterexample to this statement may be constructed along the same lines as Figure 2.

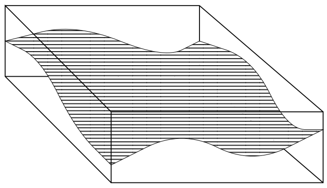

Figure 3: A disc 'spanning' a block.

6 Near-exponential decay for small p

The purpose of this section is to prove Theorem 3.2. This is an immediate consequence of the following result, which states that, in the plaquette percolation model with q close to 1, we may find with large probability a sphere lying in $[Q]^{\{1/4\}}$ and enclosing a specified point, with an appropriate bound on its radius.

Define the event

$$S(r) = \Big\{ [Q]^{\{1/4\}} \text{ contains a sphere of radius at most } r \\ \text{having the point } (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \text{ in its inside} \Big\}.$$

Theorem 6.1 *There exists $q_0 < 1$ such that, for every $q > q_0$ and for every $k = 1, 2, \ldots$, there exists $\alpha_k = \alpha_k(q) > 0$ such that*

$$P_q^{\mathbb{P}}(S(r)) \geq 1 - \exp\left(-\frac{\alpha_k r}{\lambda_k(r)} \right) \qquad \textit{for all } r \geq 0.$$

Before proving this result, we indicate why it implies Theorem 3.2.

Proof of Theorem 3.2. If $\mathrm{rad}(C(\mathcal{E}_1)) > r$, there can exist no sphere in $\mathbb{R}^3 \setminus [K]$ with O in its inside with radius r or less. Passing to the dual plaquette model via (3), we deduce that

$$P_p^{\mathbb{L}}\Big(\mathrm{rad}(C(\mathcal{E}_1)) > r\Big) \leq 1 - P_q^{\mathbb{P}}(S(r)) \quad \text{where } p + q = 1,$$

and the claim of the theorem follows from Theorem 6.1. □

The remainder of this section is devoted to proving Theorem 6.1. Since we shall be working entirely with the plaquette model, we shall abbreviate $P_q^{\mathbb{P}}$ to P_q throughout, and we shall write E_q for the associated expectation operator.

As in [11], we shall attempt to demonstrate the existence of certain surfaces of plaquettes within blocks. Such an event may be described roughly by saying that the block contains a disc lying in $[Q]^{\{1/4\}}$ which 'separates' the top and bottom faces of the block, and whose boundary is suitably 'well behaved', as illustrated in Figure 3. There follows a precise definition of the event in question, beginning with some further notation.

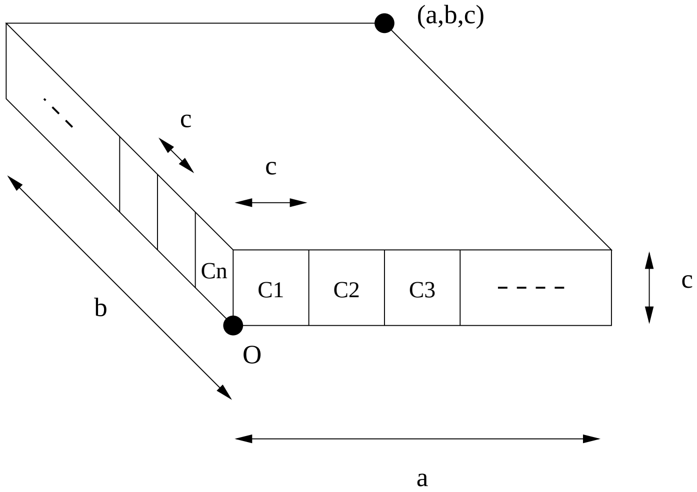

Figure 4: The discs $C_1, C_2, \ldots, C_n$ are the squares around the periphery of the block $H = H(a,b,c)$.

We define a *block* to be any subset of $\mathbb{R}^3$ of the form $[a,d] \times [b,e] \times [c,f]$; in particular, we let

$$H(a,b,c) = [0,a] \times [0,b] \times [0,c],$$

illustrated in Figure 4. Let c divide both a and b, and let $H = H(a,b,c)$. With $n = 2(a/c + b/c)$, we consider the sequence of square discs $(C_1, C_2, \ldots, C_n)$ lying in ΔH as illustrated in Figure 4. By a *loop around* H we mean a loop which is a union of the form $\nu_1 \cup \{x_{12}\} \cup \nu_2 \cup \{x_{23}\} \cup \cdots \cup \nu_n \cup \{x_{n1}\}$ where ν_i is the interior of an arc, ν_i lies in the interior of C_i, and x_{ij} is a point lying in the interior of the arc $C_i \cap C_j$. A disc *across* H is a disc $D \subseteq H$ with the properties that $D \cap \Delta H = \Delta D$, and ΔD is a loop around H. We define the set of plaquettes

$$\mathbb{P}(H) = \{u \in \mathbb{P} : \langle u \rangle \subseteq H \ , \ \langle u \rangle \not\subseteq \Delta H\},$$

and the random subset of $\mathbb{R}^3$

$$W(H) = [Q \cap \mathbb{P}(H)]^{\{1/4\}} \cap H.$$

Finally, we let $D(a,b,c)$ be the event that $W(H)$ contains a disc across H. This is the event referred to in the preceding paragraph. Note that $D(a,b,c)$ is an increasing cylinder event defined on the finite set of plaquettes $\mathbb{P}(H)$.

We shall sometimes need to refer to events corresponding to $D(a,b,c)$ but referring to a block not located at the origin, and we shall do this as follows. Let T by any bijection of $\mathbb{P}$ to itself (in the cases we shall consider, T will correspond to a translation or rotation of $\mathbb{R}^3$). Then T induces a natural bijection from Π to itself, which we shall also denote by T. For any event A we refer to the event $\{\pi \in \Pi : T(\pi) \in A\}$ as a *copy* of A.

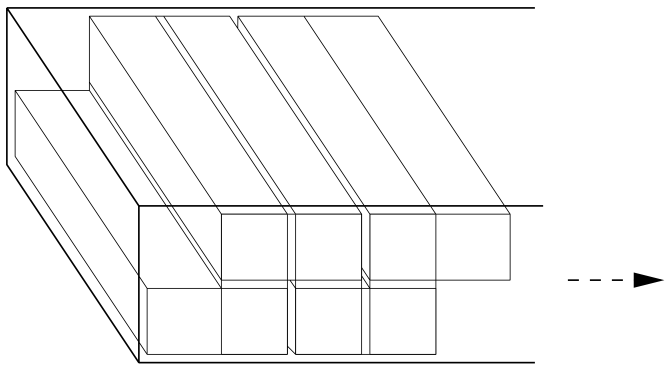

Figure 5: The construction for Lemma 6.3.

We shall now present three lemmas abstracted from [11] concerning events of the form $D(a, b, c)$. In each case we shall give a brief sketch of the proof, referring the reader to [11] for the details.

Lemma 6.2 *For any positive integers m and x we have that*

$$D(16mx, 8mx, 4mx) \supseteq B_1 \cup B_2 \cup \cdots \cup B_m$$

for certain cylinder events $B_1, B_2, \ldots, B_m$ which are copies of $D(16mx, 8mx, 4x)$ and which are defined on disjoint sets of plaquettes.

Sketch Proof. The proof involves considering a 'stack' of m disjoint blocks congruent to $H(16mx, 8mx, 4x)$ whose union is $H(16mx, 8mx, 4mx)$. The result follows because a disc across any one of the smaller blocks is also a disc across the larger block. □

Lemma 6.3 *For any positive integers m and x we have*

$$D(16mx, 8mx, 4x) \supseteq C_1 \cap C_2 \cap \cdots \cap C_n,$$

where $n = (16m - 3)^2$, and $C_1, C_2, \ldots, C_n$ are certain copies of $D(4x, 2x, x)$.

Sketch Proof. The proof is based on a construction taken from [11] which is illustrated in Figure 5; the construction involves building up a larger block from a sequence of overlapping congruent smaller blocks in different orientations. If we have a disc across each of the smaller blocks, we can construct a disc across the larger block; the topological justification for this is given in [11]. We use this construction twice: firstly we use $16m-3$ congruent copies of $H(4x, 2x, x)$ to build $H(8mx, 4x, 2x)$, and we then use $16m - 3$ congruent copies of $H(8mx, 4x, 2x)$ to build $H(16mx, 8mx, 4x)$. □

Lemmas 6.2 and 6.3 together allow us to relate the events $D(16mx, 8mx, 4mx)$ and $D(4x, 2x, x)$, these being events defined on similar blocks having dimensions in the ratio $4m : 1$. The following lemma shows how the events $D(4x, 2x, x)$ can be related to spheres enclosing the origin.

Lemma 6.4 *There exist fixed positive integers w and h such that the following holds. For every positive integer x there exist copies $A_1, A_2, \dots, A_h$ of $D(4x, 2x, x)$ such that*

$$S(wx) \supseteq A_1 \cap A_2 \cap \cdots \cap A_h.$$

Sketch Proof. One first shows that using copies of $D(4x, 2x, x)$ one may construct a disc across a block with sides in the ratio $4 : 4 : 1$ (by a construction similar to that in the preceding lemma). Six blocks of this latter shape may be used to construct a cubic shell enclosing the origin, and this shell contains a sphere provided each of the blocks contains a disc. As in the preceding result, there are some non-trivial topological details which may be found in [11]. □

For any integer $i \geq 1$ we write $D_i = D(4 \cdot 8^i, 2 \cdot 8^i, 8^i)$. Our approach to proving Theorem 6.1 will be to obtain successively stronger lower bounds on the probability of D_i (for large i), and to use Lemma 6.4 in order to translate these into bounds concerning $S(r)$.

Proposition 6.5 *There exists $q_0 < 1$ such that the following holds. If $q > q_0$, there exist $c_0 = c_0(q) > 0$ and $\sigma_0 = \sigma_0(q) \in (0, 1)$ such that*

$$1 - P_q(D_i) \leq \exp\left[-c_0(8^i)^{\sigma_0}\right] \quad \textit{for all } i \geq 1.$$

Proof. Let $i \geq 2$. Applying Lemma 6.2 in the case $m = 2$, and using the fact that the events B_1 and B_2 in the lemma are independent, we find that

$$P_q(D_i) \geq 1 - \left[1 - P_q\Big(D(4 \cdot 8^i, 2 \cdot 8^i, 4 \cdot 8^{i-1})\Big)\right]^2.$$

Applying Lemma 6.3 (with $m = 2$ and hence $n < 2^{10}$) and using the FKG inequality (see [8], page 34) we have that

$$P_q(D_i) \geq 1 - \left[1 - P_q(D_{i-1})^{2^{10}}\right]^2 \quad \text{for all } q.$$

It is the case that

$$1 - x^N \leq N(1 - x) \quad \text{if } N \geq 1 \text{ and } 0 \leq x \leq 1, \tag{4}$$

and hence

$$1 - P_q(D_i) \leq M[1 - P_q(D_{i-1})]^2,$$

where $M = 2^{20}$. Iterating the last inequality gives

$$1 - P_q(D_i) \leq M^{-1}\Big[M(1 - P_q(D_1))\Big]^{2^{i-1}},$$

and the claim of the proposition follows provided that $M(1 - P_q(D_1)) < 1$. Now, D_1 occurs provided all 32×16 plaquettes in the block $H(32, 16, 8)$ lying in one given horizontal plane are retained, and, by a simple calculation, the last inequality is satisfied provided $q \geq 1 - 2^{-29}$. □

We shall deduce Theorem 6.1 from Proposition 6.5, with the same value of q_0. We saw above that one may take $q_0 = 1 - 2^{-29}$. This value of q_0 may be decreased by a more careful application of the same ideas; for example, using the construction in [11], one may obtain $q_0 = 1 - 1/15616$. It seems unlikely that such improvements will be useful in practice, and therefore we do not seek here to minimise the value of q_0.

In order to strengthen the bound on $P_q(D_i)$, we shall require some additional probabilistic tools taken from [7]. Suppose A is an increasing cylinder event of Π, defined on a finite set of plaquettes F. Define the random variable Ψ_A to be the minimum number of plaquettes which one needs to remove from the set of retained plaquettes Q in order to prevent A from occurring; that is,

$$\Psi_A(\pi) = \min\left\{\sum_{\pi'}\Big\{\pi(f) - \pi'(f)\Big\} : \pi' \leq \pi,\ \pi' \notin A\right\}, \quad \pi \in \Pi.$$

Note that $\Psi_A(\pi) = 0$ if $\pi \notin A$.

Lemma 6.6 *If B and $C_1, C_2, \ldots, C_m$ are increasing cylinder events such that*

$$B \supseteq C_1 \cup C_2 \cup \cdots \cup C_m,$$

and $C_1, C_2, \ldots, C_m$ are defined on disjoint sets of plaquettes, then

$$\Psi_B \geq \Psi_{C_1} + \Psi_{C_2} + \cdots + \Psi_{C_m}.$$

The proof of this lemma is immediate from the definition of Ψ_A; we omit the details, which are related to those of equation (13.21) of [7].

For any event A we write

$$\Lambda_q(A) = -\log(1 - P_q(A)).$$

Lemma 6.7 *If $0 < t < u < 1$, there exists $c = c(t, u) > 0$ such that*

$$\Lambda_u(A) \geq cE_t(\Psi_A)$$

for every increasing cylinder event A.

Lemma 6.8 *If $0 < s < t < 1$, there exist $a = a(s,t) > 0$ and $b = b(s,t) > 0$ such that*

$$E_t(\Psi_A) \geq a\Lambda_s(A) - b$$

for every increasing cylinder event A.

Here are some remarks about these two lemmas, which are essentially equations (13.24) and (13.25) of [7]. Two steps are needed in order to deduce the above formulations from the latter equations. Firstly, the results in [7] apply in the more general context of random cluster measures; to obtain the specialisation to product measure we set the 'cluster-weighting factor' referred to as 'q' in [7] to unity. Secondly, we apply the inequalities of [7] not to the event A but to its complement; this complement is a decreasing event which corresponds via (3) to an increasing event in the dual bond percolation model on $\mathbb{L}_+$. For further details of the proofs of the lemmas, the reader is referred to [7].

Proof of Theorem 6.1. As mentioned earlier, our approach will be to use the above lemmas to obtain successive improvements to the bound in Proposition 6.5. We begin by explaining how Lemmas 6.6–6.8 may be applied in a general context. Suppose A, B, and C are increasing cylinder events such that

$$\begin{aligned} A &\supseteq B_1 \cup B_2 \cup \cdots \cup B_m, \\ B &\supseteq C_1 \cap C_2 \cap \cdots \cap C_n, \end{aligned} \tag{5}$$

where the B_i are copies of B defined on disjoint sets of plaquettes, and the C_j are copies of C. If $0 < s < t < u < 1$, we have the following chain of inequalities relating the probabilities of A and C. Let $a = a(s,t)$, $b = b(s,t)$, $c = c(t,u)$ be given as in Lemmas 6.7–6.8. Then

$$\begin{aligned} \Lambda_u(A) &\geq cE_t(\Psi_A) && \text{by Lemma 6.7} \\ &\geq cmE_t(\Psi_B) && \text{by (5) and Lemma 6.6} \\ &\geq cm(a\Lambda_s(B) - b) && \text{by Lemma 6.8} \\ &\geq cm\Big(-a\log[1 - P_s(C)^n] - b\Big) && \text{by (5) and the FKG inequality} \\ &\geq cm\Big(-a\log[n(1 - P_s(C))] - b\Big) && \text{by (4)} \\ &= cm\Big(a\Lambda_s(C) - a\log n - b\Big). \end{aligned}$$

We make use of this inequality by setting $A = D_i$ and $C = D_j$ where $i > j \geq 1$. With $m = 8^{i-j}/4$ and $n = (16m - 3)^2$, we have by Lemmas 6.2 and 6.3 that (5) is valid with $B = D(4 \cdot 8^i, 2 \cdot 8^i, 4 \cdot 8^j)$. Writing $I = 8^i$ and $J = 8^j$ we deduce that, if $0 < s < t < u < 1$, there exist strictly positive functions a', b', c' of s, t, u such that

$$\Lambda_u(D_i) \geq (I/J)\Big[c'\Lambda_s(D_j) - a'\log(I/J) - b'\Big]. \tag{6}$$

We may assume that $c' < 1$, whence (6) is valid whenever $i \geq j \geq 1$.

We may now use this inequality as follows: given a lower bound for the sequence $(\Lambda_q(D_i) : i \geq 1)$ which holds for all $q > q_0$, we may substitute this into the right side of (6) with a suitable choice of j, to obtain an improved bound. We shall iterate this method to obtain the conclusion of the theorem.

Let q_0 be as in Proposition 6.5, let $q_0 < u < 1$, and choose s and t such that $q_0 < s < t < u$ (for definiteness, choose q_0, s, t, u in arithmetic progression). By Proposition 6.5, we have $\Lambda_s(D_j) \geq c_0 J^{\sigma_0}$ for $j \geq 1$, where $J = 8^j$ and $\sigma_0 = \sigma_0(s)$ satisfies $0 < \sigma_0 < 1$. Let i be large, and let $j = \lfloor (2\log i)/(\sigma_0 \log 8) \rfloor$; there exists a positive constant $\gamma = \gamma(\sigma_0)$ such that

$$\gamma J \leq (\log I)^{2/\sigma_0} \leq 8\gamma J,$$

where $I = 8^i$. We substitute this value of j into (6), and deduce the existence of positive functions a'', b'', c'', c''' of s, t, u such that

$$\begin{aligned}\Lambda_u(D_i) &\geq \frac{I}{(\log I)^{2/\sigma_0}} \left(c''(\log I)^2 - a'' \log \left[\frac{I}{(\log I)^{2/\sigma_0}} \right] - b'' \right) \\ &\geq c''' \frac{I}{(\log I)^{2/\sigma_0 - 2}} \qquad \text{for all } i \geq 1.\end{aligned}$$

(The above argument yields such an inequality for large i, and the value of c''' may be chosen in order that it hold for small i also.) In conclusion, we have proved that, for every $q > q_0$, there exist $c_1 = c_1(q) > 0$ and $\sigma_1 = \sigma_1(q) > 0$ such that

$$\Lambda_q(D_i) \geq \frac{c_1 I}{(\log I)^{\sigma_1}} \quad \text{for all } i \geq 1.$$

The next step is to prove that, for each $k \geq 1$, there exist $c_k = c_k(q) > 0$ and $\sigma_k = \sigma_k(q) > 0$ such that

$$\Lambda_q(D_i) \geq \frac{c_k I}{\lambda_k(I)^{\sigma_k}} \quad \text{for all } i \geq 1. \tag{7}$$

This may be proved by induction on k, and the details are omitted. Each stage is proved via (6) using the same argument as above, but now choosing j in such a way that $J = 8^j$ behaves approximately as $(\log I)^2$.

Let $k \geq 1$ and $q > q_0$. Since $\sigma_k > 0$, inequality (7) with k replaced by $k+1$ implies that

$$\Lambda_q(D_i) \geq \frac{d_k I}{\lambda_k(I)} \quad \text{for some } d_k = d_k(q) > 0 \text{ and all } i \geq 1,$$

which we write as

$$1 - P_q(D_i) \leq \exp\left(-\frac{d_k I}{\lambda_k(I)} \right). \tag{8}$$

We now appeal to Lemma 6.4. Let w and h be given as in that lemma. For any integer $i \geq 1$, taking $x = 8^i$ and $r = w8^i$, we have that

$$\begin{aligned} 1 - P_q(S(r)) &\leq 1 - P_q(D_i)^h && \text{by Lemma 6.4 and the FKG inequality} \\ &\leq h\Big[1 - P_q(D_i)\Big] && \text{by (4)} \\ &\leq \exp\left(-\frac{d'_k r}{\lambda_k(r)}\right) \end{aligned}$$

for some $d'_k = d'_k(q) > 0$. Finally, we extend this conclusion to all r as follows. For $r \geq w$, we choose i such that $w8^i \leq r \leq w8^{i+1}$. Since $S(r) \supseteq S(w8^i)$, we have that $P_q(S(r)) \geq P_q(S(w8^i))$, and the conclusion of the theorem follows with $\alpha_k = d'_k/8$. $\square$

7 Uniqueness for large p

The purpose of this section is to prove Theorem 3.3. We start with some notation. For any set U of plaquettes, we recall the notation $[U] = \bigcup_{f \in U} \langle f \rangle$, the union of the convex hulls of the plaquettes in U. We define a *splitting set* to be a finite set of plaquettes $U \subseteq \mathbb{P}$ with the following properties.

(i) $[U]$ is a connected subset of $\mathbb{R}^3$.

(ii) $\mathbb{R}^3 \setminus [U]$ has at least one bounded connected component.

It is useful to think of a splitting set as a closed surface of plaquettes, although our definition is in fact considerably more permissive than this.

For any set $T \subseteq \mathbb{R}^3$, we define its *inside* $\mathrm{ins}(T)$ to be the union of all the bounded connected components of $\mathbb{R}^3 \setminus T$, and its *outside* to be the union of all unbounded connected components of $\mathbb{R}^3 \setminus T$. We say that a set $T \subseteq \mathbb{R}^3$ *separates* $\mathbb{R}^3$ if $\mathbb{R}^3 \setminus T$ has more than one connected component. Note that, if T is bounded, then $\mathbb{R}^3 \setminus T$ has exactly one unbounded connected component.

Our proof of Theorem 3.3 depends on the following proposition.

Proposition 7.1 *There exists $p_1 < 1$ such that, if $p > p_1$, almost surely with respect to $P_p^{\mathbb{L}}$ there exists an infinite sequence of splitting sets $U_1, U_2, \ldots$ with the following properties.*

(i) *The insides of the U_i form an increasing sequence of sets whose union is $\mathbb{R}^3$.*

(ii) *If $e \in \mathbb{L}$ is any edge such that $\langle e \rangle \subseteq [U_i]$ for some i, then e is open.*

Our proof of Proposition 7.1 depends on the following 'well known' result. Although a similar statement is proved in [5] (Lemma 2.1), we have been unable to find a published proof of the required fact. Therefore we include a proof here.

Lemma 7.2 *Let $\omega \in \Omega$ and $\pi \in \Pi$ be dual configurations of edges and plaquettes as in (3). Suppose that the connected component of $K(\omega)$ at the origin, C say, is finite. Then there exists a splitting set $U \subseteq Q(\pi)$ with $[C] + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in its inside.*

Proof. This proof is motivated by the method in the appendix of [5]. Suppose that $|C| < \infty$. Let D be the set of edges on the 'boundary' of C:

$$D = \Big\{ \{x, y\} : x \in V(C) \text{ and } y \notin V(C) \Big\},$$

and let P be the corresponding set of plaquettes

$$P = \{f(e) : e \in D\}.$$

Clearly P is a finite set, and $P \subseteq Q(\pi)$.

For a set $A \subseteq \mathbb{R}^3$ and a point $x \in \mathbb{R}^3$ we say that A *encloses* x if x lies in the inside of A. We write $h = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and we claim that $[P]$ encloses h. To see this, firstly note that, since the sets concerned are simplicial, the concepts of connectedness and path-connectedness coincide. Suppose we have a path from h to infinity; that is, a continuous mapping $\gamma : [0, \infty) \to \mathbb{R}^3$ with $\gamma(0) = h$ and whose image is unbounded. Define an *occupied cell* of C to be a set of the form $[x, x+1] \times [y, y+1] \times [z, z+1] \subseteq \mathbb{R}^3$ where x, y, z are integers, and whose centre $(x + \frac{1}{2}, y + \frac{1}{2}, z + \frac{1}{2})$ lies in $V(C) + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. By considering the 'last' point of γ lying in any occupied cell of C, we see that the image of γ must intersect $[P]$. Hence $[P]$ encloses h as required.

Let $P_1, P_2, \dots, P_k$ be subsets of P such that $[P_1], [P_2], \dots, [P_k]$ are precisely the connected components of $[P]$. We claim that some $[P_i]$ encloses h, from which the conclusion of the lemma will follow. Suppose on the contrary that no $[P_i]$ encloses h, and write $\overline{P_i} = [P_i] \cup \operatorname{ins}([P_i])$. Each $\overline{P_i}$ is closed, does not separate $\mathbb{R}^3$, and does not contain h. We claim that, for $i \neq j$, either $\overline{P_i} \cap \overline{P_j} = \emptyset$ or one of the pair $\overline{P_i}$, $\overline{P_j}$ is a subset of the other. This is proved as follows. Note first that each $\overline{P_i}$ is connected, and assume that i and j are such that $\overline{P_i} \cap \overline{P_j} \neq \emptyset$ and $i \neq j$. Firstly, if $\overline{P_i} \cap [P_j] \neq \emptyset$, then some point in $[P_j]$ lies in $\operatorname{ins}([P_i])$. Since $[P_j]$ is connected, and is disjoint from $[P_i]$, $[P_j]$ lies entirely in some bounded connected component of $\mathbb{R}^3 \setminus [P_i]$, and therefore $[P_j] \subseteq \operatorname{ins}([P_i])$. It follows that $\overline{P_j} \subseteq \operatorname{ins}([P_j])$, implying that $\overline{P_j} \subseteq \overline{P_i}$. Secondly, if $\overline{P_i} \cap [P_j] = \emptyset$ but $\overline{P_i} \cap \operatorname{ins}([P_j]) \neq \emptyset$, then a similar reasoning yields that $\overline{P_i} \subseteq \operatorname{ins}([P_j])$. In either case, the required conclusion holds.

It follows by [13] (§57; Section I, Theorem 9 and Section II, Theorem 2) that $W = \overline{P_1} \cup \overline{P_2} \cup \cdots \cup \overline{P_k}$ does not separate $\mathbb{R}^3$. Now $h \notin W$, whence h lies in the unique component of $\mathbb{R}^3 \setminus W$. This component is unbounded, and this yields a contradiction of the assumption that $[P]$ encloses h. □

Proof of Proposition 7.1. The first step is to show the following. In the plaquette percolation model with $q > 1 - p_c$, there exists $P_q^{\mathbb{P}}$-a.s. an infinite sequence of splitting sets lying in the set Q of retained plaquettes, and having property (i) of the proposition. (Note that this event is an increasing subset of Π.) This statement is proved as follows.

Let $\omega \in \Omega$ and $\pi \in \Pi$ satisfy (3), and let $n \geq 1$. Suppose that $p < p_c$. Consider the graph $K(\omega) \cup B(n)$, and let A be the connected component of this graph containing the box $B(n)$. Since $p < p_c$, A is $P_p^{\mathbb{L}}$-a.s. finite. It follows from Lemma 7.2 that there exists $P_q^{\mathbb{P}}$-a.s. a set of retained plaquettes which forms a splitting set having $B(n) + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in its inside.

If there does not exist an infinite sequence of splitting sets as claimed above, then there exists some (random) n such that $B(n) + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ lies in the inside of no splitting set of retained plaquettes. By the remark above, this occurs with probability zero, so we have proved our earlier claim.

Now let $\omega \in \Omega$. We say that a plaquette $f \in \mathbb{P}$ is *occupied* if and only if:

for each of the four edges $e \in \mathbb{L}$ such that $e \subseteq f$, we have $\omega(e) = 1$.

We write Q' for the set of occupied plaquettes. The following statements are clear:

(i) For every $f \in \mathbb{P}$, $P_p^{\mathbb{L}}(f \in Q') = p^4$.

(ii) The measure governing Q' is '1-dependent'. That is to say, if A and B are subsets of $\mathbb{P}$ such that $[A]$ and $[B]$ are disjoint, then the random sets $Q' \cap A$ and $Q' \cap B$ are independent of one another.

The following stochastic domination follows from results in [15] (see also [8], Section 7.3). Let $1 - p_c < q < 1$. If p^4 is sufficiently close to 1, the measure governing Q' stochastically dominates the product measure $P_q^{\mathbb{P}}$. That is to say, there exists $p_1 = p_1(q) < 1$ such that the measure governing Q' stochastically dominates $P_q^{\mathbb{P}}$ whenever $p \geq p_1$.

Let $1 - p_c < q < 1$ and $p \geq p_1(q)$. We deduce from the statement at the beginning of this proof that Q' contains $P_p^{\mathbb{L}}$-a.s. a sequence of splitting sets satisfying condition (i) of Proposition 7.1. Finally, it follows from the definition of an occupied plaquette that this sequence of splitting sets also satisfies condition (ii) of the proposition. □

Proof of Theorem 3.3. Let p_1 be as in Proposition 7.1; we may assume that $p_c < p_1 < 1$. Let $p > p_1$. We have that $\eta^1(p) > 0$, whence by the zero–one law K possesses $P_p^{\mathbb{L}}$-a.s. an infinite $\mathcal{E}_1$-subgraph. As noted in Section 2, it follows from the definition of $\mathcal{E}_1$ that there exists $P_p^{\mathbb{L}}$-a.s. a unique $\mathcal{E}_1$-component, which we denote I.

We claim that $I \in \mathcal{E}_0$. This implies the result of the theorem, for the following reasons. Suppose $\mathcal{E}$ is a measurable entanglement system. Since $\mathcal{E}_0 \subseteq \mathcal{E}$, I is an $\mathcal{E}$-graph, whence it lies in some infinite $\mathcal{E}$-component, denoted I', say. Since

$\mathcal{E} \subseteq \mathcal{E}_1$, such a component I' must be a subgraph of I, hence I $(= I')$ is a $\mathcal{E}$-component. Similarly, if I'' is any infinite $\mathcal{E}$-component, I'' must be a subgraph of an infinite $\mathcal{E}_1$-component, implying that $I'' \subseteq I$. Hence I is the unique infinite $\mathcal{E}$-component.

We now justify the claim that $I \in \mathcal{E}_0$. Let F be any finite subgraph of I. By Proposition 7.1, there exists $P_p^{\mathbb{L}}$-a.s. some splitting set U such that $[F]$ lies in the inside of $[U]$, and every edge $e \in \mathbb{L}$ satisfying $\langle e \rangle \subseteq [U]$ is open. Note that, for any edge e, exactly one of the following holds: (i) $\langle e \rangle \subseteq [U]$; (ii) $\langle e \rangle$ intersects the inside of $[U]$; (iii) $\langle e \rangle$ intersects the outside of $[U]$. We write B for the graph consisting of all edges $e \in \mathbb{L}$ satisfying $\langle e \rangle \subseteq [U]$, and we write A for the graph consisting of all edges $e \in I$ such that $\langle e \rangle$ intersects the inside of U. We shall show that the graph $A \cup B$ is an $\mathcal{F}$-graph, and satisfies $F \subseteq A \cup B \subseteq I$. The above claim will then follow by the definition of $\mathcal{E}_0$.

Suppose that $A \cup B$ is not an $\mathcal{F}$-graph, so that $[A \cup B]$ is separated by sphere S, say. We shall use standard topological techniques to modify S so as to obtain a sphere which separates I, giving a contradiction. For a detailed justification of the topological steps, which are similar to some of those used in [11], see for example [16]. Also, for examples of similar arguments in the topology literature, see [14], Chapter 2.

Since $[U]$ is connected, so is $[B]$, whence B lies in either the inside or the outside of S. Since S separates $[A \cup B]$, there must exist an edge $e \in A$ such that $\langle e \rangle$ lies in the opposite 'side' (inside or outside) of S from $[B]$. Consider the intersections of S and $[U]$. After deforming S by a small amount if necessary, we may assume that all such intersections are transverse. Noting that S is disjoint from $[B]$, we deduce that $S \cap [U]$ consists of finitely many disjoint loops, each one lying in the interior of $\langle f \rangle$ for some plaquette $f \in U$. (The *finiteness* is a consequence of the requirement that spheres be simplicial complexes.) We shall show that, by modifying S, we may strictly reduce the number of such loops whilst retaining the property that S separates $[B] \cup \langle e \rangle$. It will follow by induction that S may be chosen so that the number of such loops is zero.

Suppose the number of loops is non-zero. Consider the loops lying in one particular plaquette $\langle f \rangle$ as subsets of the interior of $\langle f \rangle$, and let α be an innermost loop among these. Then α bounds a disc D on $\langle f \rangle$, which has no further intersections with S or $[A \cup B]$. We apply 'surgery' to S along α as follows. Remove a thin annulus neighbourhood of α from S, and replace it with two slightly shifted copies of D lying parallel to $\langle f \rangle$, one on each side (see Figure 6). We write S' for the modified surface thus obtained from S. We make all the alterations in a sufficiently small neighbourhood to ensure that S' remains disjoint from $[A \cup D]$. We now consider the form of S'. We have effectively cut S around a loop, and attached two almost-coincident discs to the cut edges. There are two possible outcomes, depending on whether the added discs lie in the inside or the outside of S. The latter case is illustrated in Figure 7. In either case it may be seen that S' is the union of two disjoint spheres, and by considering the positions of

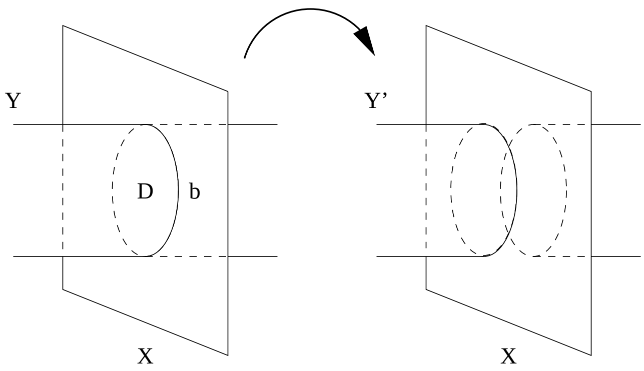

Figure 6: Removing a component of $S \cap [U]$.

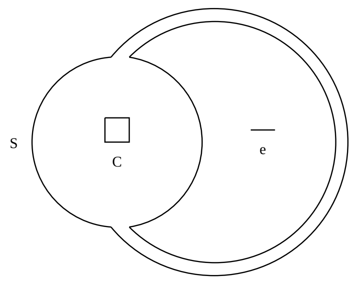

Figure 7: A cross section of the modified surface S', in one of the possible cases. In this case, the smaller of the two resulting spheres separates $[B] \cup \langle e \rangle$.

$[B]$ and $\langle e \rangle$, one of these spheres must separate $[B] \cup \langle e \rangle$ (perhaps with the roles of inside and outside reversed compared with S). We let S'' be this sphere. We note that $S'' \cap [U]$ has strictly fewer components (loops) than has $S \cap [U]$. We now repeat such surgery inductively until we obtain a sphere S_1 which separates $[A \cup B]$ and which is disjoint from $[U]$.

There are three possibilities: (i) S_1 lies in the inside of $[U]$; (ii) $[U]$ lies in the inside of S_1; (iii) each lies in the outside of the other. However, since S_1 separates $[B] \cup \langle e \rangle$, while $[B]$ is a subset of $[U]$ and $\langle e \rangle$ lies in the inside of $[U]$, it is readily seen that only the first of these three is possible. Thus we have a sphere S_1 which lies in the inside of $[U]$, and separates $[A \cup B]$. Since the graph $I \setminus A$ lies entirely in the union of $[U]$ and the outside of $[U]$, it follows that S_1 separates I. This is a contradiction since $I \in \mathcal{E}_1$. We have therefore proved as required that $A \cup B \in \mathcal{F}$.

It follows by the definition of A that $F \subseteq A$ and $A \subseteq I$. We have proved above that $A \cup B \in \mathcal{F} \subseteq \mathcal{E}_1$, and it follows by property (E2) of the entanglement system $\mathcal{E}_1$ that $A \cup B \subseteq I$. This concludes the proof that $I \in \mathcal{E}_0$. $\square$

8 Uniqueness above p_c

The purpose of this section is to prove Theorem 3.4; henceforth, we abbreviate $P_p^{\mathbb{L}}$ to P_p. The basic argument is the following. For $p > p_c$, there exists P_p-a.s. a unique infinite *connected* component C, say (see [4]). Any $\mathcal{E}_0$-component which intersects C must contain C as a subgraph, and hence if there were more than one infinite $\mathcal{E}_0$-component, at least one would necessarily have empty intersection with C. We shall show that this latter event has probability zero, using the result proved in [9] that the critical probability for percolation on a half-space of $\mathbb{L}$ is the same as that for the whole space.

Here is some notation. For $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$, we define the *half-space* $H_1(x)$ to be the subgraph of $\mathbb{L}$ consisting of all edges having at least one vertex lying in $\{x_1+1, x_1+2, \dots\} \times \mathbb{Z}^2$. Similarly, we define the half-space $H_2(x)$ by replacing $\{x_1+1, x_1+2, \dots\} \times \mathbb{Z}^2$ with $\{\dots, x_1-2, x_1-1\} \times \mathbb{Z}^2$. In a similar manner, we define the half-spaces $H_i(x)$ for $i = 3, 4, 5, 6$ to be the analogous graphs corresponding to each of the two senses of the other two coordinates.

Proof of Theorem 3.4. Let

$$Y = \{\text{there exists an infinite } \mathcal{E}_0\text{-component containing } O \text{ which has no infinite connected subgraph}\}.$$

We shall prove that $P_p(Y) = 0$ if $p > p_c$. The conclusion of the theorem will then follow by the above discussion and the almost sure uniqueness of the infinite connected component.

We shall use a 're-start argument' as follows. Define the event

(9)
$$E_n = \Big\{K \cap B(n) \text{ has an } \mathcal{F}\text{-subgraph containing } O \text{ and some vertex of } \partial B(n)\Big\}.$$

It follows from the definition of $\mathcal{E}_0$ that $|C(\mathcal{E}_0)| = \infty$ if and only if E_n occurs for infinitely many n. If $\{E_n \text{ i.o.}\}$ occurs, we choose the smallest n_1 for which E_{n_1} occurs, and try to 'grow' an infinite connected open graph starting from the vertex of $\partial B(n_1)$ referred to in (9), working in a half-space which does not intersect $B(n_1)$. If we succeed, then $C(\mathcal{E}_0)$ has an infinite connected subgraph. If not, we must fail within some finite distance, and hence we may find some larger n_2 such that E_{n_2} occurs, but we have not yet 'examined' any edges outside $B(n_2)$. We then repeat the process. At each step, there is some fixed positive probability of finding an infinite connected graph, and with sufficient care we may show that some such event occurs with probability one.

We now present the details of the argument. By a *stopping time* we mean a random variable X taking values in $\{1, 2, \dots\} \cup \{\infty\}$ such that the event $\{X \le n\}$ is a cylinder event defined in terms of the states of the edges of $B(n)$. We shall construct an increasing sequence of stopping times with the property that, if any one of them is infinite, then the event Y does not occur; we shall then prove that almost surely some stopping time is infinite.

Firstly, let

$$N_1 = \min\{n : E_n \text{ occurs}\},$$

where, here and subsequently, the minimum of the empty set is taken to be ∞. We now define M_1 as follows. If $N_1 = \infty$ we set $M_1 = \infty$. Otherwise, since E_{N_1} occurs, we may let x_1 be the first vertex of $\partial B(N_1)$ (according to some pre-determined ordering) such that $K \cap B(N_1)$ has an $\mathcal{F}$-subgraph containing O and x_1. Now, given any vertex $x \neq O$ there exists a unique n such that $x \in \partial B(n)$, and there also exists some $i = i(x) \in \{1, 2, \dots, 6\}$ such that the half-space $H_i(x)$ is disjoint from $B(n)$ (i is not always unique, but we may always choose i according to some pre-determined rule). Given $x \neq O$ and some integer $m > n$ we define the event

$$\begin{aligned} G_m(x) = \{&K \cap H_i(x) \cap B(m) \text{ has a connected subgraph containing} \\ &x \text{ and some vertex of } \partial B(m)\}, \end{aligned}$$

where i is as above. If $N_1 < \infty$, we define

$$M_1 = \min\{m > N_1 : G_m(x_1) \text{ does not occur}\}.$$

We now proceed to define N_k and M_k inductively for all k as follows. Suppose $N_1, M_1, \dots, N_k, M_k$ have been defined (where $k \geq 1$). If $M_k = \infty$ we set $N_{k+1} = \infty$, and otherwise

$$N_{k+1} = \min\{n > M_k : E_n \text{ occurs}\}.$$

If $N_{k+1} = \infty$ we set $M_{k+1} = \infty$; otherwise, as before let x_{k+1} be a vertex 'demonstrating' $E_{N_{k+1}}$ as in (9), and define

$$M_{k+1} = \min\{m > N_{k+1} : G_m(x_{k+1}) \text{ does not occur}\}.$$

The random variables $N_1, M_1, N_2, M_2, \dots$ are stopping times satisfying

$$N_1 \leq M_1 \leq N_2 \leq M_2 \leq \cdots$$

with the corresponding strict inequalities holding so long as the random variables are finite. In the following, we write (for example) '$A, B < \infty$' for the assertion that both A and B are finite. For all $k \geq 1$ we have

$$\begin{aligned} &P_p(N_1, M_1, \dots, N_k, M_k < \infty) \\ &\quad = P_p(N_1 < \infty) P_p(M_1 < \infty \mid N_1 < \infty) \cdots P_p(N_k < \infty \mid N_1, M_1, \dots, M_{k-1} < \infty) \\ &\qquad\quad \times P_p(M_k < \infty \mid N_1, M_1, \dots, M_{k-1}, N_k < \infty) \\ &\quad \leq P_p(M_1 < \infty \mid N_1 < \infty) \cdots P_p(M_k < \infty \mid N_1, M_1, \dots, M_{k-1}, N_k < \infty) \\ &\quad = (1 - \theta_H(p))^k \quad \text{for } k \geq 1, \end{aligned}$$

where $\theta_H(p)$ is the probability that $H_1(O) \cap K$ has an infinite connected subgraph containing O. It was proved in [9] that $\theta_H(p) > 0$ when $p > p_c$, and it follows from the above that

$$P_p(N_i < \infty \text{ and } M_i < \infty \text{ for all } i \geq 1) = 0.$$

We may now complete the proof by observing that if any of the above stopping times is infinite, Y cannot occur. Indeed, if for some k, $N_k = \infty$ but $M_{k-1} < \infty$, then E_n occurs for only finitely many n, whence there exists no infinite open $\mathcal{E}_0$-graph containing O. If, on the other hand, there exists k such that $M_k = \infty$ but $N_k < \infty$, then O is contained in some $\mathcal{F}$-subgraph of $K \cap B(N_k)$ which shares a vertex with an infinite connected subgraph, whence O is contained in an open $\mathcal{E}_0$-graph having an infinite connected subgraph. In either case, the event Y does not occur. □

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