Flows and ferromagnets

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July 21, 2005

Abstract

The two-point correlation function of a Potts model on a graph G may be expressed in terms of the flow polynomials of 'Poissonian' random graphs derived from G by replacing each edge by a Poisson-distributed number of copies of itself. This fact extends to Potts models the so-called random-current expansion of the Ising model.

1 Introduction

The Tutte polynomial and its relatives have rarely been distant from the work of Dominic Welsh. They play important roles in matroid theory, [20], computational complexity, [22, 23, 24], and models of statistical physics, [21, 24]. They provide the natural way to count and relate a variety of objects defined on graphs. We show here that they permit a representation of the two-point correlation function of a ferromagnetic Potts model on a graph G in terms of the flow polynomials of certain related random graphs. This representation extends to general Potts models the so-called random-current expansion for Ising models, wielded with great effect in [1, 2, 3, 16] and elsewhere, and it amplifies the links between the Potts partition function and the Tutte polynomial surveyed earlier by Welsh and Merino, [24].

Two key elements of the analysis of the Ising model on a graph G are the random-cluster representation and the random-current expansion. The former is valid for all Potts models (and more besides), but the latter has not previously been extended beyond the Ising model. It hinges on an expansion of the partition function in terms of 0/1-vectors indexed by edges and such that, for every vertex v, the sum of the values over edges incident to v is even.

Such a vector may be recognised as a 'mod-2 flow'. It turns out that the q-state Potts partition function corresponds similarly to counts of 'mod-q flows' on a graph derived from G in the following way. Let $\lambda > 0$, and replace every edge e of G by $P(e)$ parallel edges, where the $P(e)$ are independent Poisson-distributed random variables with parameter λ. The quantity of interest is the mean number of non-zero mod-q flows on the resulting random graph.

There is a powerful method of 'path-manipulation' by which many important results have been proved for the Ising model. This method has a simple form when set in the context of a Poissonian random graph, and we illustrate this in Section 5 with a version of the 'switching lemma' of [1].

A short tour of graph polynomials appears in Section 2. In Section 3 is introduced the Potts and random-cluster models, and the main result is proved in Section 4. Applications to the Ising model are summarized in Section 5. The principal open area is to extend the random-current analysis to Potts models with general q.

2 Graph polynomials

Let $G = (V, E)$ be a finite graph, possibly containing multiple edges and loops. The Whitney and Tutte polynomials of G are well known to graph theorists, and we begin with a reminder of their definitions. The (Whitney) *rank-generating function* of G was introduced in [25] and is given by

$$W_G(u, v) = \sum_{E' \subseteq E} u^{r(G')} v^{c(G')}, \qquad u, v \in \mathbb{R}, \tag{2.1}$$

where $r(G') = |V| - k(G')$ is the *rank* of the subgraph $G' = (V, E')$, and $c(G') = |E'| - |V| + k(G')$ is its *co-rank*. Here, $k(G')$ denotes the number of components of G'. Note that

$$W_G(u, v) = (u/v)^{|V|} \sum_{E' \subseteq E} v^{|E'|} (v/u)^{k(E')}, \qquad u, v \neq 0. \tag{2.2}$$

The rank-generating function has various useful properties, and it occurs in several contexts in graph theory, see [6, 19]. The Tutte (or dichromatic) polynomial of G was introduced independently in [18, 19], and may be expressed as

$$T_G(u, v) = (u - 1)^{|V|-1} W_G\big((u - 1)^{-1}, v - 1\big). \tag{2.3}$$

This also is a function of two variables. For suitable values of these variables, it provides counts of colourings, forests, and flows, and of other combinatorial quantities. The principal purpose of the current paper is to explore the use of the Whitney/Tutte polynomial in the study of the correlation functions of the Potts model, and to this end we define next the *flow polynomial* of G.

We turn G into a oriented graph by allocating an orientation to each edge $e \in E$, and we denote the resulting digraph by $\vec{G} = (V, \vec{E})$. If the edge $e = \langle u, v \rangle \in E$ is oriented from u to v, we say that f *leaves* u and *arrives* at v. It will turn out that the choices of orientations are immaterial to the principal conclusions that follow. Let $q \in \{2, 3, \dots\}$. A function $f : \vec{E} \to \{0, 1, 2, \dots, q-1\}$ is called a *mod-q flow* on $\vec{G}$ if

$$\sum_{\substack{\vec{e} \in \vec{E}: \\ \vec{e} \text{ leaves } v}} f(\vec{e}) - \sum_{\substack{\vec{e} \in \vec{E}: \\ \vec{e} \text{ arrives at } v}} f(\vec{e}) = 0 \qquad \text{modulo } q, \quad \text{for all } v \in V,$$

which is to say that flow is conserved at every vertex. A mod-q flow f is called *non-zero* if $f(\vec{e}) \neq 0$ for all $\vec{e} \in \vec{E}$. Let $C_G(q)$ be the number of non-zero mod-q flows on $\vec{G}$. It is fundamental that the quantity $C_G(q)$ does not depend on the orientations of the edges of G, and the proof may be found in [19]. The function $C_G(q)$, viewed as a function of q, is called the *flow polynomial* of G.

The flow polynomial may be obtained as an evaluation of the Whitney/Tutte polynomial with two particular parameter values, as follows:

$$\begin{aligned} C_G(q) &= (-1)^{|E|} W_G(-1, -q) \\ &= (-1)^{|E|-|V|+1} T_G(0, 1-q), \qquad q \in \{2, 3, \dots\}. \end{aligned} \tag{2.4}$$

See [6, 19]. We shall later write $C(G; q)$ for $C_G(q)$, and similarly for other polynomials when the notational need arises.

3 Potts and random-cluster models

Amongst models for ferromagnetism, the Potts model is one of the most studied. It has two principal parameters, the 'inverse temperature' $\beta \in (0, \infty)$ and the number $q \in \{2, 3, \dots\}$ of local states. When $q = 2$, the Potts model becomes the Ising model. Let $G = (V, E)$ be a finite graph which for simplicity we assume to have no loops. It is convenient to allow a separate parameter for each edge of G, and thus we let $\mathbf{J} = (J_e : e \in E)$ be a vector of non-negative numbers, and we set

$$p_e = 1 - e^{-\beta J_e q}, \qquad e \in E. \tag{3.1}$$

The configuration space of the q-state Potts model on G is the set $\Sigma = \{1, 2, \dots, q\}^V$. The Potts measure on Σ is given by

$$\pi_{\beta \mathbf{J},q}(\sigma) = \frac{1}{Z^{\mathrm{P}}} \exp\left\{ \sum_{e \in E} \beta J_e (q\delta_e(\sigma) - 1) \right\}, \qquad \sigma \in \Sigma, \tag{3.2}$$

where, for $e = \langle x, y \rangle \in E$,

$$\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y} = \begin{cases} 1 & \text{if } \sigma_x = \sigma_y, \\ 0 & \text{otherwise,} \end{cases}$$

and $Z^{\mathrm{P}} = Z^{\mathrm{P}}_G$ is the partition function

$$Z^{\mathrm{P}} = \sum_{\sigma \in \Sigma} \exp\left\{ \sum_{e \in E} \beta J_e (q\delta_e(\sigma) - 1) \right\}. \tag{3.3}$$

Since $\beta J_e \geq 0$, the Potts measure $\pi_{\beta \mathbf{J},q}$ allocates greater probability to configurations for which $\delta_e(\sigma) = 1$ for a larger set of edges e. That is, it prefers configurations in which many neighbour-pairs have the same state, and in this regard the model is termed 'ferromagnetic'.

A central quantity is the 'two-point correlation function' given by

$$\tau_{\beta \mathbf{J},q}(x, y) = \pi_{\beta \mathbf{J},q}(\sigma_x = \sigma_y) - \frac{1}{q}, \qquad x, y \in V. \tag{3.4}$$

We shall work here with $q\tau_{\beta \mathbf{J},q}(x, y) = \pi_{\beta \mathbf{J},q}(q\delta_{\sigma_x,\sigma_y} - 1)$ and, for ease of notation in the following, we write

$$\sigma(x, y) = q\tau_{\beta \mathbf{J},q}(x, y), \qquad x, y \in V, \tag{3.5}$$

thereby suppressing reference to the parameters $\beta \mathbf{J}$ and q.

Two of the most successful ways of studying the Ising/Potts models are the so-called 'random-cluster model' and the 'random-current expansion'. We define next the random-cluster model, and we explain its relevance to the Potts model. The random-current expansion for the Ising model will be reviewed in Section 5.

In the (bond) percolation model on G, each edge is declared at random to be either 'open' or 'closed'. An edge is declared 'open' with some given probability p, and closed otherwise, and different edges are allocated independent states. The percolation model is basic to the study of disordered media, particularly when the underlying graph is part of a 'crystalline' lattice such as the d-dimensional cubic lattice $\mathbb{L}^d$. See [9] for a full account. When G

is a complete graph, the percolation model is usually called an 'Erdős–Rényi random graph', see [13].

The random-cluster measure on G is obtained through a perturbation of the percolation measure, as follows. Let $\mathbf{p} = (p_e : e \in E) \in [0,1]^E$ and $q \in (0,\infty)$. The configuration space is $\Omega = \{0,1\}^E$. For $\omega \in \Omega$ and $e \in E$, we say that e is ω-*open* (or, simply, open) if $\omega(e) = 1$, and ω-*closed* otherwise. The random-cluster probability measure on Ω is defined by

$$\phi_{\mathbf{p},q}(\omega) = \frac{1}{Z^{\mathrm{RC}}}\left\{\prod_{e\in E} p_e^{\omega(e)}(1-p_e)^{1-\omega(e)}\right\}q^{k(\omega)}, \qquad \omega \in \Omega,$$

where $k(\omega)$ denotes the number of ω-open components on the vertex-set V, and $Z^{\mathrm{RC}} = Z_G^{\mathrm{RC}}$ is the appropriate normalizing factor. We sometimes write $\phi_{G,\mathbf{p},q}$ when the role of G is to be emphasized.

It is common to take $p_e = p$ for all $e \in E$, in which case we write $\phi_{p,q}$ for $\phi_{\mathbf{p},q}$. The special case $q = 1$, $\mathbf{p} = p$ is evidently the percolation measure with parameter p, in which case we write $\phi_p = \phi_{p,1}$. It turns out that the random-cluster model with $q \in \{2,3,\dots\}$ corresponds in a certain way to the Potts model on G with q local states and with $\beta\mathbf{J}$ satisfying (3.1). Specifically, the two-point correlation function of the latter is (up to a harmless factor) equal to the connection probability of the former,

$$\tau_{\beta\mathbf{J},q}(x,y) = (1-q^{-1})\phi_{\mathbf{p},q}(x \leftrightarrow y), \qquad x,y \in V, \tag{3.6}$$

where $x \leftrightarrow y$ means that there exists a path of open edges from x to y. The random-cluster model was introduced by Fortuin and Kasteleyn around 1970, and has been reviewed recently in [8, 10, 11].

The random-cluster partition function Z_G^{RC} is given by

$$Z_G^{\mathrm{RC}}(p,q) = \sum_{\omega\in\Omega} p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}q^{k(\omega)},$$

and is easily seen by (2.2) to satisfy

$$Z_G^{\mathrm{RC}}(p,q) = q^{|V|}(1-p)^{|E|}W_G\left(\frac{p}{q(1-p)}, \frac{p}{1-p}\right), \qquad p \neq 1,$$

a relationship which provides a link with other classical graph-theoretic quantities. See [5, 6, 7, 17, 24].

4 Potts correlations and flow counts

It is shown in this section that the Potts correlation functions (3.4) may be expressed in terms of flow polynomials associated with a certain 'Poissonian' random graph derived from G by replacing each edge by a random number of copies. This extends to general q the random-current expansion of the Ising model described in Section 5.

For any vector $\mathbf{m} = (m(e) : e \in E)$ of non-negative integers, let $G_{\mathbf{m}} = (V, E_{\mathbf{m}})$ be the graph with vertex set V and, for each $e \in E$, with exactly $m(e)$ edges in parallel joining the endvertices of the edge e; the original edge e is itself removed. Note that

$$|E_{\mathbf{m}}| = \sum_{e \in E} m(e). \tag{4.1}$$

Let $\boldsymbol{\lambda} = (\lambda_e : e \in E) \in [0, \infty)^E$. Let $\mathbf{P} = (P(e) : e \in E)$ be a family of independent random variables such that $P(e)$ has the Poisson distribution with parameter λ_e. The random graph $G_{\mathbf{P}} = (V, E_{\mathbf{P}})$ is called a *Poisson graph with intensity* $\boldsymbol{\lambda}$. Let $\mathbb{P}_{\boldsymbol{\lambda}}$ and $\mathbb{E}_{\boldsymbol{\lambda}}$ denote the corresponding probability measure and expectation operator.

For $x, y \in V$, $x \neq y$, we denote by $G_{\mathbf{P}}^{x,y}$ the graph obtained from $G_{\mathbf{P}}$ by adding an edge with endvertices x, y. If x and y are already adjacent in $G_{\mathbf{P}}$, we add exactly one further edge between them. Potts-correlations and flows are related by the following theorem. The function $\sigma(x, y)$ is given in (3.5).

Theorem 4.2. *Let $q \in \{2, 3, \dots\}$ and $\lambda_e = \beta J_e$. Then*

$$\sigma(x, y) = \frac{\mathbb{E}_{\boldsymbol{\lambda}}(C(G_{\mathbf{P}}^{x,y}; q))}{\mathbb{E}_{\boldsymbol{\lambda}}(C(G_{\mathbf{P}}; q))}, \qquad x, y \in V. \tag{4.3}$$

This formula is particularly striking when $q = 2$, since non-zero mod-2 flows necessarily take only the value 1. A finite graph $H = (W, F)$ is called *even* if the degree of every vertex w is even. It is trivial that $C_H(2) = 1$ if H is even, and $C_H(2) = 0$ otherwise. By (4.3), for any graph G,

$$\sigma(x, y) = \frac{\mathbb{P}_{\boldsymbol{\lambda}}(G_{\mathbf{P}}^{x,y} \text{ is even})}{\mathbb{P}_{\boldsymbol{\lambda}}(G_{\mathbf{P}} \text{ is even})}. \tag{4.4}$$

Such observations are at the heart of the random-current expansion for Ising models. See Section 5.

Theorem 4.2 may be extended via (3.6) to the random-cluster model. Assume for simplicity that every edge has the same parameter p. The proof of the following is easily derived from Theorem 4.2, and may be found in [11]. It is obtained by expressing the flow polynomial in terms of the Tutte polynomial T, and allowing q to vary continuously.

Theorem 4.5. *Let $p \in [0, 1)$ and $q \in (0, \infty)$. Let $\lambda_e = \lambda$ for all $e \in E$, where $p = 1 - e^{-\lambda q}$.*

(i) *For $x, y \in V$,*

$$(q-1)\phi_{G,p,q}(x \leftrightarrow y) = \frac{\mathbb{E}_\lambda\big((-1)^{1+|E_{\mathbf{P}}|}T(G_{\mathbf{P}}^{x,y};0,1-q)\big)}{\mathbb{E}_\lambda\big((-1)^{|E_{\mathbf{P}}|}T(G_{\mathbf{P}};0,1-q)\big)}. \tag{4.6}$$

(ii) *For $q \in \{2,3,\dots\}$,*

$$\phi_p(q^{k(\omega)}) = (1-p)^{|E|(q-2)/q} q^{|V|}\mathbb{E}_\lambda(C(G_{\mathbf{P}};q)). \tag{4.7}$$

When $q=2$, (4.7) reduces to the curiosity

$$\phi_p(2^{k(\omega)}) = 2^{|V|}\mathbb{P}_\lambda(G_{\mathbf{P}} \text{ is even}). \tag{4.8}$$

This may be simplified further. Let $\zeta(e) = P(e)$ modulo 2. It is easily seen that $G_{\mathbf{P}}$ is an even graph if and only if G_ζ is even, and that the $\zeta(e)$, $e \in E$, are independent Bernoulli variables with

$$\mathbb{P}_\lambda(\zeta(e)=1) = \tfrac{1}{2}(1-e^{-2\lambda}) = \tfrac{1}{2}p.$$

Equation (4.7) may therefore be written as

$$\phi_p(2^{k(\omega)}) = 2^{|V|}\phi_{p/2}(\text{the open graph on } V \text{ is even}). \tag{4.9}$$

Proof of Theorem 4.2. Since the parameter β appears together with the multiplicative factor J_e, we may without loss of generality take $\beta = 1$. We begin with a calculation involving the Potts partition function Z^{P} of (3.3). Let $\mathbb{Z}_+ = \{0,1,2,\dots\}$ and let $\mathbf{m} = (m_e : e \in E) \in \mathbb{Z}_+^E$. By a Taylor expansion in the variables J_e,

$$\begin{aligned}\exp\Big\{-\sum_{e\in E} J_e\Big\} Z^{\mathrm{P}} &= \sum_{m\in\mathbb{Z}_+^E}\left(\prod_{e\in E}\frac{J_e^{m_e}}{m_e!}e^{-J_e}\right)\partial^{\mathbf{m}} Z^{\mathrm{P}}\Big|_{J=0}\\ &= \mathbb{E}_{\boldsymbol{\lambda}}\left(\partial^{\mathbf{P}} Z^{\mathrm{P}}\Big|_{J=0}\right)\end{aligned} \tag{4.10}$$

where

$$\partial^{\mathbf{m}} Z^{\mathrm{P}} = \left(\prod_{e\in E}\frac{\partial^{m_e}}{\partial J_e^{m_e}}\right) Z^{\mathrm{P}}, \qquad \mathbf{m} \in \mathbb{Z}_+^E.$$

By (3.3) with $\beta = 1$,

$$\begin{aligned}
\partial^{\mathbf{m}} Z^{\mathrm{P}}\Big|_{J=0} &= \sum_{\sigma\in\Sigma} \prod_{e\in E} (q\delta_e(\sigma) - 1)^{m_e} \\
&= \sum_{\sigma\in\Sigma} \prod_{e\in E_{\mathbf{m}}} (q\delta_e(\sigma) - 1) \\
&= \sum_{\sigma\in\Sigma} \prod_{e\in E_{\mathbf{m}}} \sum_{n_e\in\{0,1\}} [-\delta_{n_e,0} + \delta_{n_e,1} q\delta_e(\sigma)] \\
&= \sum_{\mathbf{n}\in\{0,1\}^{E_{\mathbf{m}}}} \sum_{\sigma\in\Sigma} (-1)^{|\{e:\, n_e=0\}|} q^{|\{e:\, n_e=1\}|} \Bigg(\prod_{e\in E_m} \delta_e(\sigma)^{n_e} \Bigg) \\
&= \sum_{\mathbf{n}\in\{0,1\}^{E_{\mathbf{m}}}} (-1)^{|\{e:\, n_e=0\}|} q^{|\{e:\, n_e=1\}|} q^{k(m,n)}, \qquad (4.11)
\end{aligned}$$

where $k(\mathbf{m}, \mathbf{n})$ is the number of connected components of the graph obtained from $G_{\mathbf{m}}$ after deletion of all edges e with $n_e = 0$. By (2.2)–(2.4),

$$\begin{aligned}
\partial^{\mathbf{m}} Z^{\mathrm{P}}\Big|_{J=0} &= (-1)^{|E_{\mathbf{m}}|} \sum_{\mathbf{n}\in\{0,1\}^{E_{\mathbf{m}}}} (-q)^{|\{e:\, n_e=1\}|} q^{k(\mathbf{m},\mathbf{n})} \\
&= (-1)^{|E_{\mathbf{m}}|} q^{|V|} W_{G_{\mathbf{m}}}(-1, -q) \qquad (4.12) \\
&= q^{|V|} C(G_{\mathbf{m}}; q). \qquad (4.13)
\end{aligned}$$

Combining (4.10)–(4.13),

$$\exp\Bigg\{ -\sum_{e\in E} J_e \Bigg\} Z^{\mathrm{P}} = q^{|V|} \mathbb{E}_{\boldsymbol{\lambda}}(C(G_{\mathbf{P}}; q)). \qquad (4.14)$$

Let $x, y \in V$. We define the unordered pair $f = (x, y)$, and write $\delta_f(\sigma) = \delta_{\sigma_x,\sigma_y}$ for $\sigma \in \Sigma$. We have that

$$\begin{aligned}
\sigma(x, y) &= \pi_{\beta\mathbf{J},q}(q\delta_f(\sigma) - 1) \\
&= \frac{1}{Z^{\mathrm{P}}} \sum_{\sigma\in\Sigma} (q\delta_f(\sigma) - 1) \exp\Bigg\{ \sum_{e\in E} \beta J_e (q\delta_e(\sigma) - 1) \Bigg\}. \qquad (4.15)
\end{aligned}$$

By an analysis parallel to (4.10)–(4.14),

$$\begin{aligned}
\exp\Bigg\{ -\sum_{e\in E} J_e \Bigg\} \sum_{\sigma\in\Sigma} (q\delta_f(\sigma) - 1) \exp\Bigg\{ \sum_{e\in E} \beta J_e (q\delta_e(\sigma) - 1) \Bigg\} \qquad (4.16) \\
= q^{|V|} \mathbb{E}_{\boldsymbol{\lambda}}(C(G^{x,y}_{\mathbf{P}}; q)),
\end{aligned}$$

and (4.3) follows by (4.14) and (4.15). $\square$

5 Random-current expansion of the Ising model

Unlike the situation with the Potts model, there is a fairly complete analysis of the Ising model. A principal part in this analysis is played by Theorem 4.2 with $q = 2$, under the heading 'random-current expansion'. This has permitted proofs amongst other things of the exponential decay of correlations in the low-β regime on the cubic lattice $\mathbb{L}^d$ with $d \geq 2$. See [1, 2, 3]. It has not so far been possible to extend this work to general Potts models, but Theorem 4.2 could play a part in such an extension.

Let $G = (V, E)$ be a finite graph without loops as before, and set $q = 2$. We restrict ourselves here to the Ising model with $J_e = J$ for all $e \in E$, and we write $\lambda = \beta J$. By Theorem 4.2,

$$\sigma(x, y) = 2\tau_{\lambda,2}(x, y) = \frac{\mathbb{P}_\lambda(G^{x,y}_{\mathbf{P}} \text{ is even})}{\mathbb{P}_\lambda(G_{\mathbf{P}} \text{ is even})}, \qquad 0 \leq \lambda < \infty. \tag{5.1}$$

The value of such a representation will become clear during the following discussion, which is based on material in [1, 15, 16]. In advance of this, we make a remark concerning (5.1). In deciding whether $G_{\mathbf{P}}$ or $G^{x,y}_{\mathbf{P}}$ is an even graph, we need only know the numbers $P(e)$ when reduced modulo 2. That is, we can work with $\zeta \in \Omega = \{0, 1\}^E$ given by $\zeta(e) = P(e) \bmod 2$. Since $P(e)$ has the Poisson distribution with parameter λ, $\zeta(e)$ has the Bernoulli distribution with parameter

$$p' = \mathbb{P}_\lambda(P(e) \text{ is odd}) = \tfrac{1}{2}(1 - e^{-2\lambda}).$$

We obtain thus from (5.1) that

$$\sigma(x, y) = \frac{\phi_{p'}(\partial\zeta = \{x, y\})}{\phi_{p'}(\partial\zeta = \varnothing)},$$

where $\phi_{p'}$ denotes product measure on Ω with density p', and

$$\partial\zeta = \left\{ v \in V : \sum_{e:\, e\sim v} \zeta(e) \text{ is odd} \right\}, \qquad \zeta \in \Omega,$$

where the sum is over all edges e incident to v. We refer to members of $\partial\zeta$ as 'sources' of the configuration ζ.

Let $\mathbf{M} = (M_e : e \in E)$ be a sequence of disjoint finite sets (possibly empty) indexed by E, and let $m_e = |M_e|$. As noted in the last section, the vector $\mathbf{M}$ may be used to construct a multigraph $G_{\mathbf{m}} = (V, E_{\mathbf{m}})$ in which each $e \in E$ is replaced by m_e edges in parallel; we may take M_e to be the set of such edges. For $x, y \in V$, we write '$x \leftrightarrow y$ in $\mathbf{m}$' if x and y lie in the same component of $G_{\mathbf{m}}$. We define the set $\partial \mathbf{M}$ of *sources* of $\mathbf{M}$ by

$$\partial \mathbf{M} = \left\{ v \in V : \sum_{e:\, e \sim v} m_e \text{ is odd} \right\}. \tag{5.2}$$

Thus, for example, $G_{\mathbf{m}}$ is even if and only if $\partial \mathbf{M}$ is empty. From the vector $\mathbf{M}$ we construct a vector $\mathbf{N} = (N_e : e \in E)$ by deleting each member of each M_e with probability $\frac{1}{2}$, independently of all other elements. That is, we let B_i, $i \in \bigcup_e M_e$, be independent Bernoulli random variables with parameter $\frac{1}{2}$, and we set

$$N_e = \{ i \in M_e : B_i = 1 \}, \qquad e \in E.$$

We write $\mathbb{P}^{\mathbf{M}}$ for the appropriate probability measure. The following lemma is pivotal for the computations which follow.

Lemma 5.3. *Let $\mathbf{M}$ and $\mathbf{m}$ be as above. If $x, y \in V$ are such that $x \neq y$ and $x \leftrightarrow y$ in $\mathbf{m}$ then, for $A \subseteq V$,*

$$\mathbb{P}^{\mathbf{M}}\big(\partial \mathbf{N} = \{x, y\},\, \partial(\mathbf{M} \setminus \mathbf{N}) = A\big) = \mathbb{P}^{\mathbf{M}}\big(\partial \mathbf{N} = \varnothing,\, \partial(\mathbf{M} \setminus \mathbf{N}) = A \bigtriangleup \{x, y\}\big).$$

Proof. Take M_e to be the set of edges of $G_{\mathbf{m}}$ parallel to e, and assume that $x \leftrightarrow y$ in $\mathbf{m}$. Fix $A \subseteq V$. Let $\mathcal{M}$ be the set of all vectors $\mathbf{n} = (n_e : e \in E)$ with $n_e \subseteq M_e$ for all e. Let $\mathbf{p}$ be a fixed path of $G_{\mathbf{m}}$ with endpoints x, y, and consider the map $\rho : \mathcal{M} \to \mathcal{M}$ given by

$$\rho(\mathbf{n}) = \mathbf{n} \bigtriangleup \mathbf{p}, \qquad \mathbf{n} \in \mathcal{M}.$$

The map ρ is one–one, and maps $\{\mathbf{n} \in \mathcal{M} : \partial \mathbf{n} = \{x, y\},\ \partial(\mathbf{M} \setminus \mathbf{n}) = A\}$ to $\{\mathbf{n}' \in \mathcal{M} : \partial \mathbf{n}' = \varnothing,\ \partial(\mathbf{M} \setminus \mathbf{n}') = A \bigtriangleup \{x, y\}\}$. Each member of $\mathcal{M}$ is equiprobable under $\mathbb{P}^{\mathbf{M}}$, and the claim follows. $\square$

Let $\lambda \in [0, \infty)$, and recall from the last section the definition of a Poisson graph with parameter λ. The following is a fairly immediate corollary of the last theorem. Let $\mathbf{M} = (M_e : e \in E)$ and $\mathbf{M}' = (M'_e : e \in E)$ be vectors of disjoint finite sets satisfying $M_e \cap M'_f = \varnothing$ for all $e, f \in E$, and suppose that the random variables $m_e = |M_e|$, $m'_e = |M'_e|$, $e \in E$ are independent and such that, for each $e \in E$, m_e and m'_e have the Poisson distribution with parameter λ. We write $\mathbf{M} \cup \mathbf{M}' = (M_e \cup M'_e : e \in E)$, and $\mathbb{P}$ for the appropriate probability measure. The following lemma is a simplification of the so-called switching lemma of [1].

Lemma 5.4. *If $x, y \in V$ are such that $x \neq y$ and $x \leftrightarrow y$ in $\mathbf{m} + \mathbf{m}'$ then, for $A \subseteq V$,*

$$\begin{aligned}\mathbb{P}\big(\partial\mathbf{M} = \{x,y\},\, \partial\mathbf{M}' = A \mid \mathbf{M} \cup \mathbf{M}'\big) \\ = \mathbb{P}\big(\partial\mathbf{M} = \varnothing,\, \partial\mathbf{M}' = A \bigtriangleup \{x,y\} \mid \mathbf{M} \cup \mathbf{M}'\big) \qquad \mathbb{P}\text{-a.s.} \quad (5.5)\end{aligned}$$

Proof. Conditional on the sets $M_e \cup M'_e$ for $e \in E$, the sets M_e are selected by the independent removal of each element with probability $\frac{1}{2}$. The claim now follows from Lemma 5.3. □

We present two applications of Lemma 5.4 to the Ising model, as in [1]. For $\mathbf{m} = (m_e : e \in E) \in \mathbb{Z}_+^E$, let

$$\partial\mathbf{m} = \left\{ v \in V : \sum_{e:\, e \sim v} m_e \text{ is odd} \right\}, \tag{5.6}$$

as in (5.2). We write as before

$$\sigma(x,y) = 2\tau_{\lambda,2}(x,y) = \pi_{\lambda,2}(2\delta_{\sigma_x,\sigma_y} - 1), \qquad x, y \in V,$$

thereby suppressing reference to λ. By (5.1),

$$\sigma(x,y) = \frac{\mathbb{P}_\lambda(\partial\mathbf{P} = \{x,y\})}{\mathbb{P}_\lambda(\partial\mathbf{P} = \varnothing)}. \tag{5.7}$$

Let $\mathbb{Q}_A$ denote the law of $\mathbf{P}$ *conditional* on the event $\{\partial\mathbf{P} = A\}$,

$$\mathbb{Q}_A(E) = \mathbb{P}_\lambda(\mathbf{P} \in E \mid \partial\mathbf{P} = A).$$

We have need of two independent copies $\mathbf{P}_1$, $\mathbf{P}_2$ of $\mathbf{P}$ with potentially different conditionings, and thus we write $\mathbb{Q}_{A;B} = \mathbb{Q}_A \times \mathbb{Q}_B$.

Lemma 5.8. *Let $x, y, z \in V$ be distinct vertices. Then*:

(i) $\sigma(x,y)^2 = \mathbb{Q}_{\varnothing;\varnothing}(x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2)$,

(ii) $\sigma(x,y)\sigma(y,z) = \sigma(x,z)\mathbb{Q}_{\{x,z\};\varnothing}(x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2)$.

Proof. (i) By (5.7) and Theorem 5.4,

$$\begin{aligned}\sigma(x,y)^2 &= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda(\partial\mathbf{P}_1 = \{x,y\},\, \partial\mathbf{P}_2 = \{x,y\})}{\mathbb{P}_\lambda(\partial\mathbf{P} = \varnothing)^2} \\ &= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda(\partial\mathbf{P}_1 = \{x,y\},\, \partial\mathbf{P}_2 = \{x,y\},\, x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2)}{\mathbb{P}_\lambda(\partial\mathbf{P} = \varnothing)^2} \\ &= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda(\partial\mathbf{P}_1 = \partial\mathbf{P}_2 = \varnothing,\, x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2)}{\mathbb{P}_\lambda(\partial\mathbf{P} = \varnothing)^2} \\ &= \mathbb{Q}_{\varnothing;\varnothing}(x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2).\end{aligned}$$

(ii) Similarly,

$$
\begin{aligned}
&\sigma(x,y)\sigma(y,z)\\
&\quad= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda\big(\partial\mathbf{P}_1 = \{x,y\},\ \partial\mathbf{P}_2 = \{y,z\}\big)}{\mathbb{P}_\lambda(\partial\mathbf{P} = \varnothing)^2}\\
&\quad= \frac{\mathbb{P}_\lambda \times \mathbb{P}_\lambda\big(\partial\mathbf{P}_1 = \varnothing,\ \partial\mathbf{P}_2 = \{x,z\},\ x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2\big)}{\mathbb{P}_\lambda(\partial\mathbf{P} = \varnothing)^2}\\
&\quad= \frac{\mathbb{P}_\lambda(\partial\mathbf{P}_2 = \{x,z\})}{\mathbb{P}_\lambda(\partial\mathbf{P} = \varnothing)} \cdot \mathbb{P}_\lambda \times \mathbb{P}_\lambda\big(x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2 \mid \partial\mathbf{P}_1 = \varnothing,\ \partial\mathbf{P}_2 = \{x,z\}\big)\\
&\quad= \sigma(x,z)\mathbb{Q}_{\{x,z\};\varnothing}(x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2),
\end{aligned}
$$

and the proof is complete. □

Theorem (5.8)(ii) leads to an important correlation inequality known as the 'Simon inequality', [16]. Let $x, z \in V$ be distinct vertices. A subset $W \subseteq V$ is said to *separate* x and z if $x, z \notin W$ and every path from x to z contains some vertex of W.

Theorem 5.9. *Let $x, z \in V$ be distinct vertices, and let W separate x and z. Then*

$$\sigma(x,z) \le \sum_{y \in W} \sigma(x,y)\sigma(y,z).$$

Proof. By Theorem 5.8(ii),

$$
\begin{aligned}
\sum_{y\in W} \frac{\sigma(x,y)\sigma(y,z)}{\sigma(x,z)} &= \sum_{y \in W} \mathbb{Q}_{\{x,z\};\varnothing}(x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2)\\
&= \mathbb{Q}_{\{x,z\};\varnothing}\big(\big|\{y \in W : x \leftrightarrow y \text{ in } \mathbf{P}_1 + \mathbf{P}_2\}\big|\big).
\end{aligned}
$$

Assume that the event $\partial\mathbf{P}_1 = \{x,z\}$ occurs. On this event, $x \leftrightarrow z$ in $\mathbf{P}_1+\mathbf{P}_2$. Since W separates x and z, the set $\{y \in W : x \leftrightarrow y \text{ in } \mathbf{P}_1+\mathbf{P}_2\}$ is non-empty on this event. Thus its (conditional) mean size is at least one under $\mathbb{Q}_{\{x,z\};\varnothing}$, and the claim follows. □

The Ising model on the graph $G = (V,E)$ corresponds as described in Section 3 to a random-cluster measure $\phi_{p,q}$ with $q = 2$. By (3.6),

$$\sigma(x,y) = 2\tau_{\lambda,2}(x,y) = \phi_{p,q}(x \leftrightarrow y),$$

where $p = 1 - e^{-\lambda q}$ and $q = 2$. The Simon inequality may be written in the form

$$\phi_{p,q}(x \leftrightarrow z) \le \sum_{y \in W} \phi_{p,q}(x \leftrightarrow y)\phi_{p,q}(y \leftrightarrow z)$$

whenever W separates x and z. It is a curious fact that this inequality holds also when $q = 1$, see [9, 12]. One may conjecture that it holds for any $q \in [1, 2]$.

Let $d \geq 2$. The random-cluster measure $\phi_{p,q}$ on the cubic lattice $\mathbb{L}^d$ may be obtained as a weak limit (with so-called free boundary conditions) of the random-cluster measure on finite boxes Λ, as $\Lambda \uparrow \mathbb{Z}^d$. The *percolation probability* is the function θ given by

$$\theta(p, q) = \phi_{p,q}(0 \leftrightarrow \infty),$$

the probability that the origin is the endpoint of an infinite open path. The critical point is defined as

$$p_{\mathrm{c}}(q) = \sup\{p : \theta(p, q) = 0\}.$$

Let $\|\cdot\|$ be a norm on $\mathbb{Z}^d$. It has been conjectured that, for $p < p_{\mathrm{c}}(q)$, there exists $\gamma = \gamma(p, q) \in (0, \infty)$ such that $\phi_{p,q}(0 \leftrightarrow x) \leq e^{-\|x\|\gamma}$ for all $x \in \mathbb{Z}^d$. This has been proved when $q = 1, 2$ and q is sufficiently large. The Simon inequality implies the following necessary and sufficient condition for exponential decay when $q = 2$.

Theorem 5.10. *Let $q = 2$ and assume that p is such that*

$$\sum_{x \in \mathbb{Z}^d} \phi_{p,q}(0 \leftrightarrow x) < \infty.$$

There exists $\gamma = \gamma(p, q) \in (0, \infty)$ such that

$$\phi_{p,q}(0 \leftrightarrow z) \leq e^{-\|z\|\gamma}, \qquad z \in \mathbb{Z}^d.$$

The proof follows standard lines, and may be found in [11, 16] together with proofs of the following facts. There is an important extension of the Simon inequality due to Lieb, [15]. This also may be proved via the flow representation of Theorem 4.2. The Lieb inequality has an important consequence for the nature of the phase transition of the Ising model, namely the 'vanishing of the mass gap'.

Let $q = 2$ and write

$$\psi(p, q) = \lim_{n \to \infty} \left\{ -\frac{1}{n} \log \phi_{p,q}(0 \leftrightarrow \partial\Lambda_n) \right\},$$

where $\Lambda_n = [-n, n]^d$ and $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$. Note that $\psi(p, q)$ is non-increasing in p, and $\psi(p, q) = 0$ if $p > p_{\mathrm{c}}(q)$. One of the characteristics of a first-order phase transition is the (strict) exponential decay of connectivity probabilities *at* the critical point. The quantity $\psi(p_{\mathrm{c}}(q), q)$ is sometimes termed the *mass gap*.

Theorem 5.11. *Let $d \geq 2$ and $q = 2$. Then $\psi(p,q)$ decreases to 0 as $p \uparrow p_c(q)$. In particular $\psi(p_c(q),q) = 0$, that is, the mass gap equals 0.*

This was proved in [15], see also [11]. The corresponding statement is known to be false for $d \geq 2$ and $q > Q(d)$ for some sufficiently large $Q(d)$. See [11, 14]. In further use of the random-current expansion (with $q = 2$), it has been proved that $\psi(p,q) > 0$ whenever $p < p_c(q)$. See [2, 3, 4] for more details of the Ising phase transition.

6 Acknowledgement

The author acknowledges the hospitality of Cornell University, where Theorem 4.2 was proved in 1991.

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