Differential Inequalities for Potts and Random-Cluster Processes

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Abstract. Let $\theta(\mathbf{J})$ be the order parameter of a (ferromagnetic) Potts or random-cluster process with bond-variables $\mathbf{J} = (J_e : e \in K)$. We discuss differential inequalities of the form

$$\frac{\partial\theta}{\partial J_e} \le \alpha(\mathbf{J})\frac{\partial\theta}{\partial J_f} \quad \text{for all } e, f \in K.$$

Such inequalities may be established for all random-cluster processes that satisfy the FKG inequality, possibly in the presence of many-body interactions (subject to certain necessary and sufficient conditions on the sets of interactions). There are (at least) two principal consequences of this. First, for a process having 'inverse-temperature' β, the critical value $\beta_c = \beta_c(\mathbf{J})$ is a strictly monotone function of $\mathbf{J}$. Secondly, at any fixed point $\mathbf{J}$ lying on the critical surface of the process, the critical exponent of θ in the limit as $\mathbf{J}' \downarrow \mathbf{J}$ is independent of the direction of approach of the limit. Such a conclusion should be valid for other critical exponents also; this amounts to a small amount of rigorous universality.

1. Introduction

Let us consider a general spin-system on $\mathbb{Z}^d$ having a finite set $\mathbf{J} = (J_e : e \in K)$ of parameters. One general approach to the question of establishing macroscopic properties of the system is to derive and analyse inequalities involving the partial derivatives of the order parameter $\theta(\mathbf{J})$. The purpose of this paper is to survey inequalities of the form

$$\frac{\partial\theta}{\partial J_e} \le \alpha(\mathbf{J})\frac{\partial\theta}{\partial J_f} \quad \text{for all } e, f \in K, \tag{1}$$

where α is a continuous function that is finite on (at least) the interior of the space of all vectors $\mathbf{J}$. (Note that (1) should, strictly speaking, be replaced by the corresponding inequalities for the finite-box approximations θ_Λ of θ, where α is independent of the choice of the box Λ.)

Two points at issue are:

(a) How may one derive inequalities (1)?

(b) How may such inequalities be used?

Inequalities of the type (1) appear to have been established first by Menshikov [12] in the special setting of bond percolation with two types of edges. The basic idea of [12] was isolated and refined by Aizenman and Grimmett [1], and discussed there in some detail for percolation and Ising models with pair-interactions. More recently, Bezuidenhout, Grimmett, and Kesten [3] have set such work in the general context of Potts and random-cluster processes. The general case of many-body interactions is treated by Grimmett [10]. In Sections 3 and 4 of the present paper, we present the basic ideas of the derivation of (1) for percolation and random-cluster processes, respectively.

Just as important as the derivation of the inequalities (1) are their uses. There appear to be two principal uses of such inequalities, and each concerns the nature of the phase transition. The first application is to the question of the strict monotonicity of the critical point as a function of the underlying parameters (for example, the strict monotonicity of the critical inverse-temperature $\beta_c = \beta_c(\mathbf{J})$, as a function of $\mathbf{J}$). The second application is to the universality of certain critical exponents. These applications of (1) are discussed in the next section.

2. Applications of the Differential Inequalities

Assume for the sake of definiteness that the vector $\mathbf{J}$ of parameters satisfies $\mathbf{J} \in (0,\infty)^K$, and that the phase transition is indicated by the order parameter θ changing from the value 0 to being strictly positive. The subcritical and supercritical regions may be defined respectively as follows. For $\mathbf{J} \in (0,\infty)^K$, let

$$\beta_c(\mathbf{J}) = \sup\big\{\beta : \theta(\beta\mathbf{J}) = 0\big\}, \tag{2}$$

and define the *subcritical* and *supercritical regions* by

$$\mathrm{SB} = \big\{\mathbf{J} : \beta_c(\mathbf{J}) > 1\big\}, \tag{3}$$

$$\mathrm{SP} = \big\{\mathbf{J} : \beta_c(\mathbf{J}) < 1\big\}. \tag{4}$$

The *critical surface* is the set C given by

$$\mathrm{C} = \big\{\mathbf{J} : \beta_c(\mathbf{J}) = 1\big\}. \tag{5}$$

The question of strict monotonicity of critical points may be cast in this framework as follows: do there exist $\mathbf{J}, \mathbf{J}' \in \mathrm{C}$ such that $\mathbf{J} \leq \mathbf{J}'$ but $\mathbf{J} \neq \mathbf{J}'$? The following argument may be made rigorous (see [1, 3]). Since C is (approximately) a contour of the function

θ, the normal to C at a point $\mathbf{J}$ lies in the direction of the gradient vector $\nabla\theta$ at $\mathbf{J}$. If (1) holds, then there exists δ (> 0) such that this gradient vector satisfies

$$\frac{\partial \theta}{\partial J_e} \geq \delta |\nabla \theta| \quad \text{for all } e \in K. \tag{6}$$

The tangent vectors to C at $\mathbf{J}$ are perpendicular to $\nabla\theta$, and it follows from (6) that every unit tangent vector has all components bounded away from zero. By considering the geometry of C, it follows that if $\mathbf{J} \in$ C and $\mathbf{J}' \geq \mathbf{J}$, $\mathbf{J}' \neq \mathbf{J}$, then $\mathbf{J}'$ lies in the interior of SP.

In the usual setting, there is a single parameter β, the 'inverse-temperature', and the hamiltonian H of the system satisfies $H(\beta, \mathbf{J}, \sigma) = H(1, \beta\mathbf{J}, \sigma)$, where σ is a configuration of spins. For given $\mathbf{J}$, the critical value is defined as in (2). Applying the arguments sketched above, one finds that $\beta_c(\mathbf{J})$ is strictly monotone in $\mathbf{J}$, in that

$$\beta_c(\mathbf{J}) > \beta_c(\mathbf{J}') \quad \text{whenever } \mathbf{J}' \geq \mathbf{J} \text{ and } \mathbf{J}' \neq \mathbf{J}, \tag{7}$$

so long as inequalities (1) *hold.* Further discussion may be found in [1].

A second application of (1) is to the question of the universality of critical exponents. It is generally believed that $\theta(\mathbf{J}')$ behaves roughly as $|\mathbf{J}' - \mathbf{J}|^b$ in the limit as $\mathbf{J}' \downarrow \mathbf{J} \in$ C; here b is a universal critical exponent, supposed to depend only on the type of the process (percolation, Ising, etc.) and the number of its dimensions. Suppose then that, for $\mathbf{J} \in$ C and for any unit vector $\mathbf{e}$ of $\mathbb{R}^K$, there exists a number $\beta_{\mathbf{J}}(\mathbf{e})$ such that

$$\theta(\mathbf{J} + \epsilon\mathbf{e}) - \theta(\mathbf{J}) \approx \epsilon^{\beta_{\mathbf{J}}(\mathbf{e})} \quad \text{as } \epsilon \downarrow 0; \tag{8}$$

the relation '$\approx$' should be interpreted in some reasonable way. As stated above, it is believed that $\beta_{\mathbf{J}}(\mathbf{e})$ is independent of $\mathbf{J}$ and of $\mathbf{e}$ (so long as $\mathbf{e} \geq \mathbf{0}$, say), and furthermore $\beta_{\mathbf{J}}(\mathbf{e})$ is believed to depend only on the type and on the number of dimensions of the process.

Suppose now that (1) holds. Since the left-hand side of (8) may be obtained by integrating $\mathbf{e}.\nabla\theta$, it may be shown that $\beta_{\mathbf{J}}(\mathbf{e})$ does not depend on the choice of $\mathbf{e}$ (so long as $\mathbf{e} \geq \mathbf{0}$, say, though actually less suffices). That is to say, at any fixed point $\mathbf{J}$ on the critical surface of the process, the critical exponent of θ does not depend on the direction of approach of $\mathbf{J}$. This is a (rather small) piece of rigorous universality, valid whenever inequalities (1) hold (or rather, corresponding inequalities for the finite-box approximations of θ).

It is valuable to note that $\beta_{\mathbf{J}}(\mathbf{e})$ is independent of $\mathbf{e}$ at any point $\mathbf{J}$ in a neighbourhood of which (1) is valid (with α finite). This is especially interesting if such a $\mathbf{J}$ lies on the boundary of the parameter space $[0, \infty)^K$. Suppose for example that $\mathbf{J}$ is such that $J_f = 0$ for some $f \in K$, and that (1) holds in a neighbourhood of $\mathbf{J}$. Then the critical exponent $\beta_{\mathbf{J}}(\mathbf{e})$ satisfies, in particular,

$$\beta_{\mathbf{J}}(\mathbf{e}_f) = \beta_{\mathbf{J}}(\mathbf{e}_g) \quad \text{for all } g \in K, \tag{9}$$

where $\mathbf{e}_g$ is the unit vector $(\delta_{eg} : e \in K)$, δ_{eg} being the Kronecker delta. Now $\beta_{\mathbf{J}}(\mathbf{e}_g)$, for $g \neq f$, is a critical exponent of a process with $J_f = 0$, *i.e.*, a process in which the interaction indexed by f is absent; whereas, $\beta_{\mathbf{J}}(\mathbf{e}_f)$ is an exponent of the 'full' process. An example of this observation is given at the end of the next section.

Since working on the boundary of the parameter space $[0,\infty)^K$ usually corresponds to 'switching off' certain interactions, special care is needed in checking the validity of (1) (with finite α) at such boundary points.

3. Percolation

The following simple example is illustrative of the derivation of inequalities (1). Consider bond percolation on the triangular lattice $\mathbb{T}$. We write $e\|f$ if e and f are parallel edges, and we denote by η_1, η_2, and η_3 the equivalence classes of the relation $\|$. Edges in η_i are declared *open* with probability p_i, independently of the states of all other edges; the three parameters of the process are $\mathbf{p} = (p_1, p_2, p_3)$. It is known that the critical surface of the process is the set of all $\mathbf{p}$ satisfying $\phi(\mathbf{p}) = 0$ where

$$\phi(\mathbf{p}) = p_1 + p_2 + p_3 - p_1p_2p_3 - 1; \tag{10}$$

see [9, 15]; this fact is completely irrelevant to the following discussion. The order parameter is the probability $\theta(\mathbf{p})$ that the origin $\mathbf{0}$ belongs to an infinite open cluster, *i.e.*,

$$\theta(\mathbf{p}) = P_{\mathbf{p}}(\mathbf{0} \leftrightarrow \infty) \tag{11}$$

where $P_{\mathbf{p}}$ is the associated probability measure. Russo's formula (see [9]) provides a representation for the partial derivatives $\partial\theta/\partial p_i$ of the form

$$\frac{\partial\theta}{\partial p_i} = E_{\mathbf{p}}(N_i), \tag{12}$$

where $E_{\mathbf{p}}(N_i)$ is the mean number of edges of η_i which are 'pivotal' for the event $\{\mathbf{0} \leftrightarrow \infty\}$; we recall that an edge is 'pivotal' for an event A if the state of this edge determines whether or not A occurs. (Actually, Russo's formula implies (12) with θ replaced by a 'finite-box approximation' $\theta_\Lambda(\mathbf{p}) = P_{\mathbf{p}}(\mathbf{0} \leftrightarrow \partial\Lambda)$, where $\partial\Lambda$ is the boundary of a cube Λ containing the origin, and with N_i replaced by the number $N_i(A)$ of edges that are pivotal for the event $A = \{\mathbf{0} \leftrightarrow \partial\Lambda\}$.) Therefore, the 'macroscopic' derivative $\partial\theta/\partial p_i$ may be expressed as the sum,

$$\frac{\partial\theta}{\partial p_i} = \sum_{e\in\eta_i} P_{\mathbf{p}}\big(e \text{ is pivotal for } \{\mathbf{0} \leftrightarrow \infty\}\big) \tag{13}$$

of probabilities of events having a local character.

Now each edge e ($\in \eta_i$) has a 'bottom left' endvertex, which is also the 'bottom left' endvertex of a unique edge in η_j (where $j \neq i$); call this latter edge e'. It is not hard to see that there exists a continuous function $\alpha(\mathbf{p})$, finite on $(0,1)^3$, such that

$$P_{\mathbf{p}}\big(e \text{ is pivotal for } \{\mathbf{0} \leftrightarrow \infty\}\big) \leq \alpha(\mathbf{p}) P_{\mathbf{p}}\big(e' \text{ is pivotal for } \{\mathbf{0} \leftrightarrow \infty\}\big), \tag{14}$$

implying by (13) that

$$\frac{\partial\theta}{\partial p_i} \le \alpha(\mathbf{p})\frac{\partial\theta}{\partial p_j} \tag{15}$$

as required for (1). Furthermore, α may be chosen in such a way that (15) holds for all i and j. Inequality (14) is obtained in the following way. Suppose that ω is a configuration for which e is pivotal for the event $\{\mathbf{0} \leftrightarrow \infty\}$. By making changes to the states of edges near to e, *i.e.*, within some fixed distance R of e, we may obtain a new configuration ω' for which e' is pivotal for $\{\mathbf{0} \leftrightarrow \infty\}$. Some geometrical considerations are relevant in making such local changes, but it is not difficult to see roughly how this may be done. Since ω and ω' differ only on a bounded number of edges, we have that $P_{\mathbf{p}}(\omega) \le \gamma(\mathbf{p})P_{\mathbf{p}}(\omega')$ for some $\gamma(\mathbf{p})$ which is continuous and finite on $(0,1)^3$, and which is independent of the choice of e. Summing over all such ω, we obtain that

$$\begin{aligned} P_{\mathbf{p}}\big(e \text{ pivotal for } \{\mathbf{0} \leftrightarrow \infty\}\big) &= \sum_{\omega} P_{\mathbf{p}}(\omega) \le \gamma(\mathbf{p}) \sum_{\omega} P_{\mathbf{p}}(\omega') \\ &\le N\gamma(\mathbf{p}) \sum_{\omega'} P_{\mathbf{p}}(\omega') \\ &= N\gamma(\mathbf{p}) P_{\mathbf{p}}\big(e' \text{ pivotal for } \{\mathbf{0} \leftrightarrow \infty\}\big), \end{aligned}$$

where $N = N(R)$ is a uniform upper bound for the number of configurations ω which give rise to a given ω'. To make the above argument rigorous, one replaces the event $\{\mathbf{0} \leftrightarrow \infty\}$ by the event $\{\mathbf{0} \leftrightarrow \partial\Lambda\}$ where Λ is a finite box containing the origin; the function γ may be taken to be independent of the choice of Λ.

With care, one may see that (15) holds for all $\mathbf{p} \in [0,1)^3$ and for a function $\alpha(\mathbf{p})$ which is finite on the set S of all vectors $\mathbf{p} \in [0,1)^3$ having at least two non-zero components. Note that setting $p_3 = 0$ corresponds to working on the square lattice $\mathbb{Z}^2$, rather than on the triangular lattice $\mathbb{T}$.

Having established (15), and therefore (1), we obtain information about the phase transition, as follows. Suppose that the critical surface C has equation $\phi(\mathbf{p}) = 0$ (the fact that ϕ may be taken according to (10) in this case is completely immaterial to the current discussion). One learns first that C has no 'flat sections', in the sense that if $\phi(\mathbf{p}) = 0$, and $\mathbf{p}' \ge \mathbf{p}$ but $\mathbf{p}' \ne \mathbf{p}$, then $\phi(\mathbf{p}') \ne 0$. Secondly, one obtains that, if the critical exponents $\beta_{\mathbf{p}}(\mathbf{e})$, given by

$$\theta(\mathbf{p} + \epsilon\mathbf{e}) - \theta(\mathbf{p}) \approx \epsilon^{\beta_{\mathbf{p}}(\mathbf{e})} \quad \text{as } \epsilon \downarrow 0, \text{ for } \mathbf{p} \in \mathrm{C},$$

exist, then $\beta_{\mathbf{p}}(\mathbf{e})$ does not depend on the choice of $\mathbf{e}$ (so long as $\mathbf{e} \ge \mathbf{0}$, say, though actually less suffices). Furthermore, this invariance of $\beta_{\mathbf{p}}(\mathbf{e})$ is valid for all $\mathbf{p} \in S$. A conclusion of a particularly interesting type is obtained at the point $\mathbf{p} = (\frac{1}{2}, \frac{1}{2}, 0)$, which lies on the critical surface by virtue of the fact that bond percolation on $\mathbb{Z}^2$ has critical probability $\frac{1}{2}$. With this choice of $\mathbf{p}$, we obtain the equality of the critical exponents associated with the two quantities $\theta(\frac{1}{2}, \frac{1}{2}, \epsilon)$ and $\theta(\frac{1}{2}+\epsilon, \frac{1}{2}+\epsilon, 0)$, in the limit as $\epsilon \downarrow 0$. Striking is the fact that the first is an exponent associated with the triangular lattice, and the second is associated with the square lattice. None of the above argument is based on the special structure of the pair $(\mathbb{Z}^2, \mathbb{T})$. Similar conclusions are valid for mixed percolation, with 'bonds' which may be general finite sets of vertices; such processes are hypergraphs, rather than graphs.

The argument leading to (15) was set in a general context in [1], where it was proved that any 'essential enhancement' of a percolation process changes its critical point *strictly*. Other applications of this idea are presented in [1].

Conclusions related to the above discussion of critical exponents have been reached also by Wierman (1992, private communication), using different means. The method described here may be applied also to certain other exponents, such as the one usually denoted by γ.

4. Potts and Random-Cluster Processes

The relationship between Potts and random-cluster processes was established and explored by Fortuin and Kasteleyn, and reviewed more recently by others (see [5, 6, 7, 11, 2, 4, 13, 14]). Fortuin and Kasteleyn observed that the magnetization of a ferromagnetic Potts model, with q states available at each vertex ($q \geq 2$), could be expressed in terms of the 'percolation probability' of a certain process on the edge-set of the lattice. This representation for the Potts model enables one to translate questions concerning long-range correlations in Potts models into geometrical questions for random-cluster processes. Furthermore, the random-cluster measures may be defined for any positive real q, rather than for integers only; therefore, in studying random-cluster processes, one may obtain results in a more general setting than for Potts models only.

We omit the details of the derivation of the random-cluster representation of Potts models, but we refer the reader to [2, 4, 10].

Let $G = (V, E)$ be a finite graph, and let $\widetilde{\mathbf{p}} = (\widetilde{p}_e : e \in E)$ be a vector of numbers satisfying $0 \leq \widetilde{p}_e \leq 1$, and indexed by the edge-set of G. The random-cluster measure ϕ is a probability measure on the configuration space $\Omega_E = \{0,1\}^E$ given by

$$\phi(\omega) = \frac{1}{Z}\left\{\prod_{e \in E} \widetilde{p}_e^{\,\omega(e)}(1-\widetilde{p}_e)^{1-\omega(e)}\right\} q^{k(\omega)}, \quad \omega \in \Omega_E, \tag{16}$$

where $k(\omega)$ is the number of components of the graph $(V, \eta(\omega))$, and $\eta(\omega) = \{e : \omega(e) = 1\}$ is the set of edges which are 'open' in the configuration ω. The constant Z is the appropriate normalizing factor. Note that ϕ may be expressed as product-measure together with a 'derivative' proportional to $q^{k(\omega)}$. It may be seen that ϕ satisfies the FKG inequality if and only if $q \geq 1$ (see [2, 8]).

The first step towards establishing (1) for random-cluster processes is to represent the derivatives of $\phi(A)$ in a useful way, for certain events A of interest. The following notably simple formula is valid: for any event A, we have that

$$\frac{\partial}{\partial \widetilde{p}_e}\phi(A) = \frac{1}{\widetilde{p}_e(1-\widetilde{p}_e)}\operatorname{cov}(I_e, I_A), \tag{17}$$

where I_e is the indicator function of the event $\{\omega(e) = 1\}$, and I_A is the indicator function of the event A; of course, 'cov' is the covariance operator associated with ϕ.

This formula, easily derived from (16) with hindsight, is a generalization of Russo's formula (see [9], particularly Theorem (2.33)).

We may express the covariance in (17) as

$$\operatorname{cov}(I_e, I_A) = \phi(A)\Big\{\phi\big(\omega(e) = 1 \,\big|\, A\big) - \phi\big(\omega(e) = 1\big)\Big\}. \tag{18}$$

Suppose now that $q \geq 1$ (so that ϕ satisfies the FKG inequality), and that A is an increasing event (in that, if $\omega \in A$ and $\omega \leq \omega'$, then $\omega' \in A$). In this case, the right-hand side of (18) is non-negative, and it is useful to express the difference therein as a single non-negative object, as follows. One may construct a Markov chain $(X, Y) = (X_t, Y_t)_{t \geq 0}$ taking values in $\Omega_E \times A$, such that
(a) X is irreducible with stationary measure $\phi(\cdot)$,
(b) Y is irreducible with stationary measure $\phi(\cdot \mid A)$,
(c) $X_t \leq Y_t$ for all t.
This chain may be used to represent (17), via (18), as

$$\frac{\partial}{\partial \widetilde{p}_e}\phi(A) = \frac{\phi(A)}{\widetilde{p}_e(1-\widetilde{p}_e)} \lim_{t\to\infty} P\big(X_t(e) = 0,\ Y_t(e) = 1\big), \tag{19}$$

where P is the appropriate probability measure on the sample paths of (X, Y). This formula is central to the method; it is valid for all increasing A, and whenever $q \geq 1$.

The random-cluster measure on the infinite lattice $\mathbb{Z}^d$ is constructed as follows. Let K be a finite set of vertices, not including the origin $\mathbf{0}$, and let $\mathbf{p} = (p_x : x \in K)$ be a vector of numbers satisfying $0 \leq p_x \leq 1$. As edge-set of $\mathbb{Z}^d$, we take the set $\mathbb{E} = \{e_{u,x} : u \in \mathbb{Z}^d, x \in K\}$, where $e_{u,x}$ denotes an edge with endpoints u and $u + x$. Thus, each vertex u 'interacts' with all vertices of the form $u - x$ or $u + x$, for $x \in K$. We assume that

(20) if $x \in K$, then $-x \notin K$,

(21) the graph $(\mathbb{Z}^d, \mathbb{E})$ is connected.

Let Λ be a finite box of $\mathbb{Z}^d$ containing the origin, and let $\partial\Lambda$ be the boundary of Λ, *i.e.*, $\partial\Lambda$ is the set of all vertices u ($\in \Lambda$) for which there exists x ($\in K$) such that either $u+x \notin \Lambda$ or $u-x \notin \Lambda$. We write $\mathbb{E}_\Lambda$ for the set of edges (in $\mathbb{E}$) both of whose endpoints lie in Λ, and we denote by Ω_Λ the subset of $\Omega = \{0,1\}^{\mathbb{E}}$ containing configurations ω satisfying $\omega(e) = 1$ for all $e \notin \mathbb{E}_\Lambda$. We define the 'finite-box' probability measure on Ω_Λ by

$$\phi_{\Lambda,\mathbf{p},q}(\omega) = \frac{1}{Z_\Lambda}\left\{\prod_{e\in E} p_e^{\omega(e)}(1-p_e)^{1-\omega(e)}\right\} q^{k(\omega)}, \quad \omega \in \Omega_\Lambda, \tag{22}$$

where $p_e = p_x$ if $e = e_{u,x} \in \mathbb{E}$. Note that $\phi_{\Lambda,\mathbf{p},q}$ is the measure ϕ given by (16), where G is obtained from $(\mathbb{Z}^d, \mathbb{E})$ by identifying all vertices in $\Lambda \setminus \partial\Lambda$, and $\widetilde{\mathbf{p}}$ is given by $\widetilde{p}_e = p_e$.

Suppose $q \geq 1$. It may be shown that the weak limit

$$\phi_{\mathbf{p},q} = \lim_{\Lambda\uparrow\mathbb{Z}^d} \phi_{\Lambda,\mathbf{p},q} \tag{23}$$

exists. Let $A = \{\mathbf{0} \leftrightarrow \infty\}$, the event that the origin is in an infinite component of the graph $(\mathbb{Z}^d, \eta(\omega))$, and define

$$\theta_\Lambda(\mathbf{p}, q) = \phi_{\Lambda,\mathbf{p},q}(A). \tag{24}$$

It is not hard to see that

$$\theta_\Lambda(\mathbf{p}, q) \downarrow \theta(\mathbf{p}, q) \quad \text{as } \Lambda \uparrow \mathbb{Z}^d,$$

where

$$\theta(\mathbf{p}, q) = \phi_{\mathbf{p},q}(A).$$

The function θ is the order parameter of the process, and the subcritical, supercritical, and critical regions may be defined by (2)–(5) with $\mathbf{J}$ replaced by $\mathbf{p}$.

Making use of (19), one finds that

$$\frac{\partial \theta_\Lambda}{\partial p_x} = \frac{\theta_\Lambda(\mathbf{p}, q)}{p_x(1-p_x)} \sum_{u\in\Lambda} \lim_{t\to\infty} P\big(X_t(e_{u,x}) = 0,\ Y_t(e_{u,x}) = 1\big), \tag{25}$$

if $0 < p_x < 1$, where (X, Y) is a certain bivariate Markov chain. This formula plays the role that (13) played for percolation. Fix $x, y \in K$ with $x \neq y$, and suppose that $q > 1$ (the following argument fails if $q = 1$). By following the sample paths of the process (X, Y), and by estimating the transition intensities of the chain, we may find a continuous function $\gamma(\mathbf{p}, q)$ such that

$$P\big(X_t(e_{u,x}) = 0,\ Y_t(e_{u,x}) = 1\big) \le \gamma(\mathbf{p}, q) P\big(X_t(e_{u,y}) = 0,\ Y_t(e_{u,y}) = 1\big) \tag{26}$$

for all $u \in \Lambda \backslash \partial\Lambda$. Taking into account some special effects when $u \in \partial\Lambda$, one concludes from (25) and (26) that

$$\frac{\partial \theta_\Lambda}{\partial p_x} \le 2\gamma(\mathbf{p}, q) \frac{\partial \theta_\Lambda}{\partial p_y} \quad \text{for all } x, y \in K, \tag{27}$$

as required for (1).

It is interesting to note that such arguments and conclusions are valid (with some changes) in four related situations. First, one may show that (27) holds (with finite γ) at certain points on the boundary of the parameter space $(0, 1)^K$. Suppose, for example, that K is partitioned as $K = K' \cup R$, in such a way that (20) and (21) hold when K is replaced by K'. Then γ may be taken to be finite at any point $\mathbf{p}$ satisfying $0 \le p_x < 1$ for $x \in R$, $0 < p_x < 1$ for $x \in K'$.

Secondly, inequalities (27) may be derived in the context of many-body interactions rather than pair-interactions only. In this generalization of the random-cluster process, the edges are replaced by hyperedges, being finite sets of vertices having cardinality two or more. See [10].

Thirdly, the method is valid also if one weakens the assumption that the underlying graph $(\mathbb{Z}^d, \mathbb{E})$ is vertex-transitive, replacing it by an appropriate but general assumption of periodicity. Certain extra geometrical complications arise in this general setting, especially when the interactions are many-body rather than pair. Nevertheless, one may arrive at a geometrical condition for any given interaction which is necessary and sufficient for this interaction to contribute in a vital way to the phase transition. See [10].

Finally, it may interest probabilists to note that inequalities of the type (27) may be established when the underlying measure ϕ in (16) is replaced by something of the form

$$\phi(\omega) = \left\{ \prod_{e \in E} \widetilde{p}_e^{\,\omega(e)} (1 - \widetilde{p}_e)^{1-\omega(e)} \right\} \rho(\omega), \quad \omega \in \Omega_E,$$

where ρ is a 'Radon–Nikodym' derivative satisfying certain suitable conditions.

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