DIRECTED PERCOLATION AND RANDOM WALK

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ABSTRACT. Techniques of 'dynamic renormalization', developed earlier for undirected percolation and the contact model, are adapted to the setting of directed percolation, thereby obtaining solutions of several problems for directed percolation on $\mathbb{Z}^d$ where $d \geq 2$. The first new result is a type of uniqueness theorem: for every pair x and y of vertices which lie in infinite open paths, there exists almost surely a third vertex z which is joined to infinity and which is attainable from x and y along directed open paths. Secondly, it is proved that a random walk on an infinite directed cluster is transient, almost surely, when $d \geq 3$. And finally, the block arguments of the paper may be adapted to systems with infinite range, subject to certain conditions on the edge probabilities.

1. Introduction

'Block arguments' constitute a fundamental technique for studying disordered spatial processes. For many years, physicists have appealed to the theory of renormalization, although difficulties have emerged in making such arguments mathematically rigorous. Block arguments have been widely used since [22], at least, and have led to proofs of several theorems of substance (see [3, 4, 9, 17, 27]). For a recent account of dynamic and static renormalization in the context of undirected percolation, the reader is referred to Chapter 7 of [14].

The adaptation of such methodology to directed models, such as directed percolation, is not totally straightforward. New difficulties arise through the presence of orientations on the edges of the underlying lattice, and new ideas are needed to overcome these problems. Various specific problems have arisen in the work of other authors on directed percolation. These problems may be dealt with by suitable block arguments, and the target of this paper is to show how this may be done.

We introduce directed percolation in Section 2, and we state our main results in Section 3. We proceed in Section 4 to describe the block construction which enables a comparison between a supercritical process and another directed percolation process with density close to 1. In the manner already explored in [4, 7],

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this implies the absence of infinite open paths in the critical case, as well as the continuity of the percolation probability.

As consequences of the above comparison theorem, we shall present positive answers to the following two questions posed by others. Let $d \geq 2$, let p_c be the critical probability of directed percolation on $\mathbb{Z}^d$, and suppose $p > p_c$. We say that a vertex x is *connected to infinity* if there exists an infinite open path which is directed away from its endvertex x. It is proved that, almost surely, for all vertices x, y which are connected to infinity, there exists a third vertex z, also connected to infinity, such that x and y are connected by directed open paths to z. This answers a question of Itai Benjamini, and is a cousin of the 'unique infinite cluster' theorem of [1, 8, 13].

Our second application concerns the classification of a random walk on the set of vertices attainable from the origin along directed open paths. It was proved in [16] that random walk on the infinite open cluster of undirected percolation is almost surely transient in three or more dimensions. A similar result was proved in [5] for directed percolation in three dimensions, whenever the edge density is sufficiently large. The methods used in the latter paper are quite different from those used in [16], and have other applications also. We present in Theorem 3 a positive answer to a question posed in [5], namely whether the transience result for directed percolation may be extended to all values of p satisfying $p > p_c$.

There is only little difficulty in extending results about 'nearest neighbour' directed percolation to systems with *finite* range. There has been recent interest [21, 29] in systems with *infinite* range, and particularly in whether or not they may possess an infinite open path *at* the critical point. Such systems are explored in Section 7, where it is explained how the block construction of Section 4 may be adapted to infinite-range systems satisfying some weak conditions of regularity.

Directed percolation is closely related to the contact model, for which block arguments have been used to prove results related to some of those described above (see [7, 12, 23, 24]). Although the arguments of [7] are in part useful for the present work, the discreteness of the underlying lattice leads to some special problems for directed percolation. Just as in the case of the contact model, the comparison theorem of the current paper may be used to establish further results such as a shape theorem, a complete convergence theorem, and the continuity of the critical points of slabs in the limit of large slabs. We do not present the details of the necessary proofs; an interested reader may refer to the earlier papers cited in [7], where closely related material is studied.

2. Notation

Let $\mathbb{Z}^d$ denote the set of all d-vectors $x = (x_1, x_2, \ldots, x_d)$ of integers. For $x, y \in \mathbb{Z}^d$, we define

$$|x - y| = \sum_{i=1}^{d} |x_i - y_i|.$$

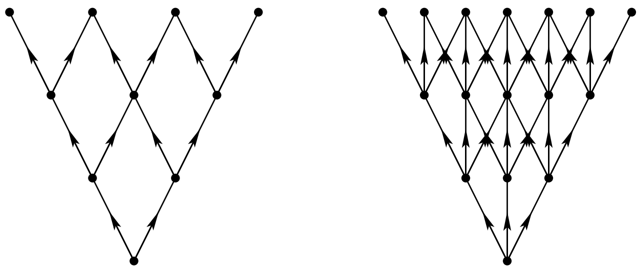

Figure 1. The graphs $\vec{\mathbb{L}}^2$ and $\vec{\mathbb{L}}^2_{\text{alt}}$.

We refer to vectors in $\mathbb{Z}^d$ as *vertices*, and we turn $\mathbb{Z}^d$ into a graph by adding an (undirected) edge $\langle x, y\rangle$ between every pair x, y of vertices such that $|x-y| = 1$. The resulting graph is denoted $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$. The *origin* of this graph is the vertex $0 = (0, 0, \ldots, 0)$. We write $x \le y$ if $x_i \le y_i$ for $1 \le i \le d$. We may use the lattice $\mathbb{L}^d$ to generate a multiplicity of directed graphs of which two feature in this paper.

Conventional model. The edge $\langle x, y\rangle$ with $x \le y$ is assigned an arrow from x to y. We write $\vec{\mathbb{L}}^d = (\mathbb{Z}^d, \vec{\mathbb{E}}^d)$ for the ensuing directed graph.

Alternative model. Each vertex $x = (x_1, x_2, \ldots, x_d)$ may be expressed as $x = (\mathbf{x}, t)$ where $\mathbf{x} = (x_1, x_2, \ldots, x_{d-1})$ and $t = x_d$. Consider the directed graph with vertex set $\mathbb{Z}^d$ and with a directed edge joining two vertices $x = (\mathbf{x}, t)$ and $y = (\mathbf{y}, u)$ whenever $\sum_{i=1}^{d-1} |y_i - x_i| \le 1$ and $u = t + 1$. We write $\vec{\mathbb{L}}^d_{\text{alt}} = (\mathbb{Z}^d, \vec{\mathbb{E}}^d_{\text{alt}})$ for the ensuing directed graph, and we note that every vertex has out-degree $1 + 2(d-1)$.

The graphs $\vec{\mathbb{L}}^2$ and $\vec{\mathbb{L}}^2_{\text{alt}}$ are sketched in Figure 1. Until recently, the conventional model has been considered the natural habitat of directed percolation in d dimensions, whereas recent results of van der Hofstad and Slade [21, 29] concerning the scaling limit of critical directed percolation in high dimensions have indicated the relevance of the alternative model. It is reasonable to think that results available for either of these models may be derived for the other also, but some technical difficulties may arise in justifying this statement in concrete examples, owing to the fact that the automorphism groups of $\vec{\mathbb{L}}^d$ and $\vec{\mathbb{L}}^d_{\text{alt}}$ are different. We shall in this paper concentrate on the alternative model, since we avoid thus certain minor complications involving periodicity. Our arguments are easily adapted to the conventional model. In either case, we write $[x, y\rangle$ for an edge which is directed from x to y. *We assume henceforth that we are studying the directed graph* $\vec{\mathbb{L}}^d_{\text{alt}}$ *where* $d \ge 2$.

In Section 7, we shall consider a generalization of the alternative model in which random directed edges of long range are added to $\mathbb{Z}^d$, rather then merely between nearest neighbours of $\vec{\mathbb{L}}^d_{\text{alt}}$. It turns out that, subject to certain natural assumptions on the parameters of such a process, the techniques developed below and elsewhere may be adapted successfully to such a model. Such results have potential applications to the work reported in [21, 29], where it is proved that the scaling limit of critical directed percolation is, for high dimension, the process known as integrated super-Brownian excursion.

Given a directed or undirected graph $G = (V, E)$, the configuration space for percolation on G is the set $\Omega = \{0,1\}^E$. For $\omega \in \Omega$, we call an edge $e \in E$ *open* if $\omega(e) = 1$ and *closed* otherwise. With Ω we associate the σ-field $\mathcal{F}$ of subsets generated by the finite-dimensional cylinders. For $0 \le p \le 1$, we let $\mathbb{P}_p$ be product measure on $(\Omega, \mathcal{F})$ with density p. There is a natural partial order on Ω given as follows: for $\omega_1, \omega_2 \in \Omega$, we write $\omega_1 \le \omega_2$ if $\omega_1(e) \le \omega_2(e)$ for all $e \in E$. We shall consider primarily percolation on the graph $\vec{\mathbb{L}}^d_{\text{alt}}$, and we suppose henceforth that $\Omega = \{0,1\}^{\vec{\mathbb{E}}^d_{\text{alt}}}$.

Let $\omega \in \Omega$. An *open path* is an alternating sequence $x_0, e_0, x_1, e_1, x_2, \dots$ of distinct vertices x_i and open edges e_j such that $e_i = [x_i, x_{i+1}\rangle$ for all i. If the path is finite, it has two endvertices x_0, x_n, and is said to connect x_0 to x_n. If the path is infinite, it is said to connect x_0 to infinity. A vertex x is said to be *connected* to a vertex y, written $x \to y$, if there exists an open path connecting x to y. If $S \subseteq \mathbb{Z}^d$, we write $x \to y$ in S if there exists an open path from x to y using only vertices contained in S. For $A, B \subseteq \mathbb{Z}^d$, we say that A is *connected to* B if there exist $a \in A$ and $b \in B$ such that $a \to b$; in this case, we write $A \to B$. We say that A is *fully connected to* B if, for all $b \in B$, there exists $a \in A$ such that $a \to b$; in this case, we write $A \to_{\text{fc}} B$. Note that, if $A \to_{\text{fc}} B$ in C, then it is necessarily the case that $B \subseteq C$. If a vertex x is connected to infinity, we write $x \to \infty$.

For $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, we write

$$C_x = C_x(\omega) = \{y \in \mathbb{Z}^d : x \to y\},$$

and we abbreviate C_0 to C. The set C_x is called the *open cluster at* x. The *percolation probability* is defined as the function

$$\theta(p) = \mathbb{P}_p(0 \to \infty) = \mathbb{P}_p(|C| = \infty).$$

Let $\psi(p) = \mathbb{P}_p(x \to \infty \text{ for some } x \in \mathbb{Z}^d)$. It is a consequence of the zero–one law that $\psi(p)$ takes values 0 and 1 only, and that $\theta(p) > 0$ if and only if $\psi(p) = 1$; cf. [14], Theorem (1.11). We define the *critical probability*

$$p_{\text{c}} = \sup\{p : \theta(p) = 0\}.$$

The cluster C_x has been defined as the set of vertices to which x is connected. We shall at some point want to think of C_x as a directed graph rather than just a set of vertices, and this is achieved by adding to C_x all open directed edges having both endvertices in C_x. The resulting graph is denoted $\vec{C}_x$. The undirected graph obtained from $\vec{C}_x$ by deleting the orientations is denoted C_x, and it will be clear from the context whether C_x is to be interpreted as a set of vertices or as a graph.

3. Principal results

Our first principal result is a re-affirmation of a theorem of [7]. The latter paper studied the contact model rather than directed percolation, but included some remarks on the extension of the results therein to directed percolation.

Theorem 1. *Let $d \geq 2$. We have that $\theta(p_c) = 0$.*

The corresponding fact for undirected percolation was proved in [3, 4, 17], and for the contact model in [7].

One of the famous theorems of undirected percolation is the statement that the infinite open cluster, when it exists, is almost surely unique (see [1, 8, 13]). There follows a version of this result for directed percolation, in answer to a question posed in a personal communication by Itai Benjamini. In order to state this in sufficient generality for later use, we introduce the usual coupling of processes for different values of p (see [14], p. 11). Let $\{U_e : e \in \vec{\mathbb{E}}^d_{\rm alt}\}$ be independent random variables with the uniform distribution on $[0, 1]$. A realization of the U_e is a vector $\eta \in [0, 1]^{\vec{\mathbb{E}}^d_{\rm alt}}$, and we define

$$\eta_p(e) = \begin{cases} 1 & \text{if } \eta(e) < p, \\ 0 & \text{otherwise.} \end{cases}$$

We call the edge e *p-open* if $\eta_p(e) = 1$, and *p-closed* otherwise. We write $\stackrel{p}{\to}$ for the relation $\to$ applied to the configuration η_p (that is, for example, $x \stackrel{p}{\to} y$ if there exists a directed path in η_p from x to y).

Theorem 2. *Let $d \geq 2$.*
(a) *Let $p_c < \alpha \leq \beta \leq 1$. For all $x, y \in \mathbb{Z}^d$,*

$$\mathbb{P}\Big(\exists z \text{ such that } x \stackrel{\alpha}{\to} z \stackrel{\alpha}{\to} \infty \text{ and } y \stackrel{\beta}{\to} z \,\Big|\, x \stackrel{\alpha}{\to} \infty,\, y \stackrel{\beta}{\to} \infty\Big) = 1.$$

(b) *The function θ is continuous on the interval* $[0, 1]$.

Part (a) is reminiscent of results of [19, 28], and part (b) of [7, 17]. One may obtain a quantification of part (a) which includes a lower bound on the probability that such a z exists within a given distance of x and y, but we do not pursue this here.

We turn now to random walk. Let G be a countably infinite connected graph with finite vertex degrees, and let 0 be a specified vertex of G. We assume for the sake of definiteness that G has neither loops nor multiple edges. Consider a random walk on the vertex set of G, that is, a sequence $X_0, X_1, \dots$ of vertices such that, for each n, X_{n+1} is chosen uniformly at random from the neighbours of X_n, each such choice being independent of all earlier choices. Since G is connected, the recurrence/transience of the random walk does not depend on the choice of initial vertex X_0. We say the G is *transient* if the random walk is transient, and we call G *recurrent* otherwise. Initiated by the results of paper [16], several authors have considered the question of whether or not an infinite open graph generated by a three-dimensional percolation model is almost surely transient. Results for undirected percolation include [16, 18, 20], and the directed case has been studied in [5] using the method of 'unpredictable paths'. Our third theorem answers a question posed in this last paper.

Theorem 3. *Let $d = 3$, and let $p > p_c$. On the event $\{|C| = \infty\}$, the undirected graph C is almost surely transient.*

It is a near triviality that C is recurrent in the corresponding statement for two dimensions, since two-dimensional lattices are recurrent graphs (this well-known fact is a consequence, for example, of the results in Section 8.4 of [10] or Lemma 7.5 of [30]). It is the case that transience holds in all dimensions $d \geq 3$; the proof of this would be similar, and is not included here.

The remaining sections contain proofs of these theorems, followed in Section 7 by a discussion of long- and infinite-range systems. Proofs are not given in their entirety, since this would be unduly long and would involve a considerable amount of duplication of material already published in [3, 4, 7, 17]. Instead, we include only the extra arguments necessary for the present setting.

4. The block construction, and proof of Theorem 1

A rigorous renormalization is the principal method introduced in [3, 4] and developed further in [7, 17]. The methods of these papers may be adapted to directed percolation more or less as indicated explicitly in [7], and we summarise this in this section. Full details are omitted, since this would involve a considerable duplication of material; the reader is referred to [7] at salient points. For clarity of exposition, we assume throughout that $d = 3$. The case $d = 2$ is easier, proceeding by path-intersection properties not valid in higher dimensions, and the more general case $d \geq 3$ may be treated by extending the current notation as described in [7].

Let K and L be positive integers, and write

$$B(L) = [-L, L]^2 \cap \mathbb{Z}^2, \quad \partial B(L) = B(L) \setminus B(L-1).$$

We refer to a box $B_{L,K} = B(L) \times [0, K]$ as a 'space–time box' of $\mathbb{Z}^3$ in which $B(L)$ plays the role of space, and the final component in $[0, K]$ plays the role of time.

The region $B_{L,K}$ has a *top* and *sides* given respectively as $B(L) \times \{K\}$ and $\partial B(L) \times [0, K]$. The top may be expressed as the union of four squares of side length L and indexed in some arbitrary manner with the set $\{-1, +1\}^2$. The sides of $B_{L,K}$ are the union of four 'facets' each of which is the union of two rectangles of side-length L and height K. We index these ensuing sub-facets in some arbitrary manner with the set $\{-1, +1\}^3$.

Let $r \geq 1$, and let $D_r = [-r, r]^2 \times \{0\}$, a 'disk' centred at the origin. Any translate of D_r is termed an *r-disk*. Let $N_t^{\mathbf{u}}(L, K)$ be the number of vertices x

in the subsquare of the top of $B_{L,K}$ indexed $\mathbf{u}$ for which $D_r \to x$ in $B_{L,K}$. Let $N_{\mathrm{s}}^{\mathbf{v}}(L,K)$ be the number of vertices x in the sub-facet of the sides of $B_{L,K}$ indexed $\mathbf{v}$ such that $D_r \to x$ in $B_{L,K}$. The subscripts 't' and 's' stand for 'top' and 'sides'.

Suppose that p is such that $\theta(p) > 0$, and let $\epsilon > 0$. By a standard argument (see [7], p. 1470), there exists an integer r such that

$$\mathbb{P}_p(D_r \to \infty) > 1 - \tfrac{1}{2}\epsilon^{12} \tag{4.1}$$

and we fix this value of r henceforth. Cf. [7], equation (6).

Let α be the minimum of: (i) the probability that 0 is fully connected to $D_r + re_3$, and (ii) the probability that 0 is fully connected to $D_r + re_1 + 2re_3$ by paths using edges contained in $(D_r + re_1) + [0, 2r]e_3$; here, e_i denotes a unit vector of the lattice in the ith direction. Let M be large enough to ensure that in M or more independent trials of an experiment having success probability α, the probability of obtaining at least one success exceeds $1-\epsilon$. Let N be large enough to ensure that, in any subset of $\mathbb{Z}^3$ having size N or larger, there exist at least M points all pairs of which are L^∞-distance at least $3r+1$ apart.

There follows the main lemma; cf. Lemma (7) of [7].

Lemma 4.1. *There exist positive integers L, K such that, for every index $\mathbf{u} \in \{-1,+1\}^2$ and every index $\mathbf{v} \in \{-1,+1\}^3$,*

$$\mathbb{P}_p\big(N_{\mathrm{t}}^{\mathbf{u}}(L,K) \ge N\big) \ge 1-\epsilon, \quad \mathbb{P}_p\big(N_{\mathrm{s}}^{\mathbf{v}}(L,K) \ge N\big) \ge 1-\epsilon.$$

Proof. This follows very closely that of Lemma (7) of [7], and we omit almost all details. The only complication arises as remarked on p. 1473 of [7]. Pick R sufficiently large that, in RN independent trials with success probability p, the chance of at least N successes exceeds $1 - \frac{1}{8}\epsilon^4$. We now follow the argument of [7] with the difference that, instead of requiring that the number $N_{\mathrm{t}}(L,K)$ of points on the top of $B_{L,K}$ which are joined to D_r by directed paths of $B_{L,K}$ satisfies $N_{\mathrm{t}}(L,K) \ge 4N$, we require instead that $N_{\mathrm{t}}(L,K) \ge 4RN$. We derive the corresponding version of [7], equation (12), with $1-\epsilon^4$ replaced by $1 - \frac{7}{8}\epsilon^4$. As in [7], p. 1473, we choose $S = S(L,M)$ such that

$$\mathbb{P}_p\big(N_{\mathrm{t}}(L,S) \ge 4RN\big) < 1 - \tfrac{7}{8}\epsilon^4 \le \mathbb{P}_p\big(N_{\mathrm{t}}(L,S-1) \ge 4RN\big). \tag{4.2}$$

By the choice of R, we have that

$$\begin{aligned}\mathbb{P}_p\big(N_{\mathrm{t}}(L,S) \ge 4N\big) &\ge (1 - \tfrac{7}{8}\epsilon^4)\mathbb{P}_p\big(N_{\mathrm{t}}(L,S-1) \ge 4RN\big)\\ &\ge (1 - \tfrac{7}{8}\epsilon^4)(1 - \tfrac{1}{8}\epsilon^4) \ge 1 - \epsilon^4.\end{aligned}$$

We now follow the argument of [7], using the left inequality of equation (4.2), in order to obtain an inequality corresponding to equation (16) of [7].

Note that the proof of the step corresponding to equation (15) of [7] is easier in the current setting, owing to the discreteness of the time variable. ∎

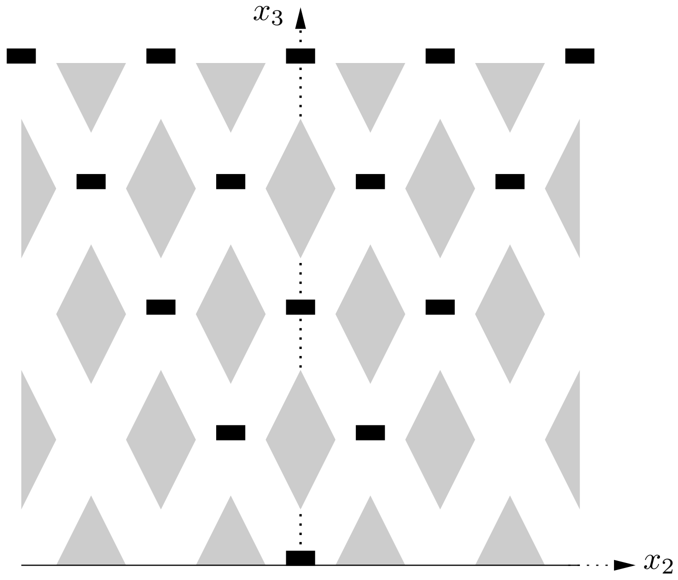

Figure 2. The target zones are drawn in black, and the region $\mathcal{R}$ in white.

Theorem 1 may now be proved exactly as was Theorem (1) of [7]. The idea is to use the block $B_{L,K}$ of Lemma 4.1 and to iterate the construction therein in order to build, with large probability, a directed path within a certain tube of $\mathbb{Z}^3$. As described in [7], this enables a stochastic comparison with a certain 1-dependent percolation model with density which may be made close to 1 by an appropriate choice of ϵ. Since the events in Lemma 4.1 depend on the states of only a finite number of edges, their probabilities are continuous functions of p. It follows that the resulting block construction is infinite with strictly positive probability, for some p' satisfying $p' < p$. Thus, if $\theta(p_c) > 0$, then $\theta(p') > 0$ for some $p' < p$, and this contradiction implies Theorem 1. We make the required construction slightly more explicit as follows.

Recall the construction of Lemmas (18)–(21) of [7]. We set $k = 11$ and $\eta > 0$; later we shall choose η to be small. Let $\epsilon > 0$ be such that $(1-\epsilon)^{4k} > 1-\eta$. With this value of ϵ, we choose r, L, K as in Lemma 4.1 and the preceding discussion, and we set $S = K + 2r$. Let $\mathcal{R}^{\pm} = [-2L, 2L] \times V^{\pm}$ where

$$V^{\pm} = \left\{ (x_2, x_3) \in \mathbb{Z}^2 : \ 0 \le x_3 \le (2k+2)S, -5L \pm \frac{L}{2S} x_3 \le x_2 \le 5L \pm \frac{L}{2S} x_3 \right\}.$$

We now define the *target zones* $V_{i,j}$ by

$$V_{i,j} = w_{i,j} + [-L, L] \times [-2L, 2L] \times [0, 2S]$$

where $w_{i,j} = k(0, iL, 2jS)$ for $i, j \in \mathbb{Z}$ with $j \ge 0$ and $i + j$ even. Finally, let

$$\mathcal{R} = \bigcup_{\substack{j \ge 0 \\ i+j \text{ even}}} \left\{ (\mathcal{R}^+ \cup \mathcal{R}^-) + w_{i,j} \right\}.$$

These regions are illustrated in Figure 2.

The usual block variables are defined as follows. Note that $D_r \subseteq V_{0,0}$. We turn our attention to the target zones $V_{-1,1}$ and $V_{1,1}$ and define indicator variables $\Xi_{-1,1}, \Xi_{1,1}$ by: $\Xi_{i,1} = 1$ if and only if some vertex in D_r is fully connected to some r-disk centred in $V_{i,1}$, by paths contained entirely within the region $\mathcal{R}$. If $\Xi_{i,1} = 1$, we let $\Delta_{i,1}$ be an earliest r-disk centred in $V_{i,1}$ with the above property ('earliest' in order of third coordinate value).

We have by Lemma 4.1, the arguments of [7], and the FKG inequality, that

$$\mathbb{P}_p\left(\Xi_{-1,1} = \Xi_{1,1} = 1\right) \geq (1-\epsilon)^{4k} > 1 - \eta.$$

The argument is now iterated from generation to generation. Having constructed $\{\Xi_{i,j} : j \leq J\}$, we find the $\Xi_{i,J+1}$ by beginning with one of the r-disks $\Delta_{i-1,J}$, $\Delta_{i+1,J}$ already found in the construction yielding $\Xi_{i-1,J}$, $\Xi_{i+1,J}$ (if both of these equal 0, we set $\Xi_{i,J+1} = 0$). If these variables both equal 1, we choose the leftmost r-disk Δ of these two disks, and we declare $\Xi_{i,J+1} = 1$ if and only if some point of Δ is connected inside $\mathcal{R}$ to every point of some r-disk centred within the target zone $V_{i,J+1}$. If $\Xi_{i,j} = 1$, we say that the target zone $V_{i,j}$ has been *achieved* from the disk D_r.

This construction enables a comparison between the original process and a 1-dependent directed conventional site percolation process on $\mathbb{Z}^2$ with a certain intensity $1 - \eta$. If η is sufficiently small, the latter process contains an infinite directed path with strictly positive probability and therefore so does the initial process.

We have shown that a supercritical directed percolation process dominates, in a way made specific above, a *two-dimensional site* percolation process with density close to 1. When proving Theorem 3 in Section 6, we shall require the stronger property that it dominates a certain *three-dimensional bond* percolation process. This will require some extra arguments.

We terminate this section with a note about the block construction for the conventional percolation model on $\vec{\mathbb{L}}^d$. The automorphism group of $\vec{\mathbb{L}}^d$ is rather different from that of $\vec{\mathbb{L}}^d_{\mathrm{alt}}$, and this has impact on the shape of the region corresponding to $B_{L,K}$ in Lemma 4.1, and on the geometry of the ensuing block construction. These turn out to be minor matters, and the results corresponding to Theorem 1–3 are valid in this setting.

5. Proof of Theorem 2

We begin with a subsidiary lemma. For $x \in \mathbb{Z}^d$, we denote by $x(r)$ the r-disk $x + D_r$.

Lemma 5.1. *Let $d \geq 2$, $p > p_c$, and $\xi > 0$. There exists a positive integer $R = R(p, \xi)$ such that*

$$\mathbb{P}_p\left(\exists z \text{ such that } x(r) \to_{\mathrm{fc}} z(r),\ y(r) \to_{\mathrm{fc}} z(r) \to \infty\right) > 1 - \xi$$

for all $r \geq R$ and all distinct $x, y \in \mathbb{Z}^d$.

Proof. This may easily be shown when $d = 2$, using path-intersection properties, and we therefore restrict ourselves to the case $d = 3$ since this contains all the ingredients sufficient for the general case $d \geq 3$.

Let $\xi > 0$, and find $\rho \in (0,1)$ such that a directed site percolation process on $\vec{\mathbb{L}}^2$ with density ρ is infinite with probability at least $1 - \frac{1}{2}\xi$. (An account of the basic properties of directed percolation may be found in Section 1.6 of [14].) Set $k = 11$ as in Section 4, and choose $\epsilon > 0$ such that:

$$2k\epsilon < \tfrac{1}{2}\xi \tag{5.1}$$

and the block process referred to at the end of Section 4 has density at least $1 - \eta > \sqrt{\rho}$. With this choice of ϵ, we choose R according to equation (4.1).

Beginning with the disk $x(R)$, we make some initial steps of the block constructions of Section 4, but in a direction which carries us away from $y(R)$. More specifically, we may assume without loss of generality that

$$x_1 - y_1 = \max\{|x_i - y_i| : 1 \leq i \leq 2\}, \tag{5.2}$$

and for the moment we assume in addition that,

$$\begin{cases} \text{for all } u \in y + [-L, L] \times [-2L, 2L] \times [0, 2S] \text{ and} \\ v \in x + (kL, 0, 2kS) + [-L, L] \times [-2L, 2L] \times [0, 2S], \\ \vec{\mathbb{L}}^3_{\text{alt}} \text{ possesses a directed path from } u \text{ to } v. \end{cases} \tag{5.3}$$

We shall see later how to adapt the proof when (5.3) fails.

As described in Section 4, with probability at least $(1 - \epsilon)^{2k} > 1 - \frac{1}{2}\xi$, $x(R)$ is connected to every point in some R-disk centred in the region $x + (kL, 0, 2kS) + [-L, L] \times [-2L, 2L] \times [0, 2S]$ by paths contained within the 'tube' $x + [-2L, 2L] \times V^+$. Let A be the event that such an R-disk $w(R)$ can be found, and we pick the earliest in the ordering induced by third coordinate value.

Assume that the event A occurs. We now rotate our frames of reference and use the disks $y(R)$ and $w(R)$ to initiate block constructions within disjoint subsets of $\mathbb{Z}^3$, namely $y + \mathcal{R}$ and $w + \mathcal{R}$ where $\mathcal{R}$ is given in Section 4. Since the difference in the first coordinates of y and the centre of $w(R)$ exceeds $(k-1)L > 9L$, and since the depth of $\mathcal{R}$ is $4L$, this may be done.

Each step in the block constructions from $y(R)$ and $w(R)$ is successful with probability at least $\sqrt{\rho}$, whence *both* are successful with probability at least $(\sqrt{\rho})^2 = \rho$, which by assumption exceeds p_c. It follows that the set of (i,j) such that both $V_{i,j}(y)$ and $V_{i,j}(w(R))$ are achieved from $y(R)$ and $w(R)$ respectively is infinite with probability at least $1 - \frac{1}{2}\xi$.

We call the pair (i,j) *green* if, for all $u \in V_{i,j}(y)$ and $v \in V_{i,j}(w(R))$, $u \to v$ in the convex hull of $V_{i,j}(y)$ and $V_{i,j}(w(R))$. We have by assumption (5.3) that

$$\gamma = \mathbb{P}_p\big((i,j) \text{ is green}\big) > 0,$$

and it follows by an application of the FKG inequality that, with probability (conditional on A) at least $1 - \frac{1}{2}\xi$, there exists a green (i, j) such that: $V_{i,j}(y)$ is achieved from $y(R)$ and $V_{i,j}(w(R))$ is achieved from $w(R)$, and in addition the block constructions from $y(R)$ and $w(R)$ include infinite paths of blocks beginning respectively at $V_{i,j}(y)$ and $V_{i,j}(w(R))$.

If A occurs, and also the last event, there exists z such that $x(R) \to_{\mathrm{fc}} z(R) \to \infty$ and $y(R) \to_{\mathrm{fc}} z(R)$. The probability of failure does not exceed $1 - \mathbb{P}_p(A) + \frac{1}{2}\xi < \xi$, and the claim is proved.

Finally we return to assumption (5.3). When (5.3) fails, we need to continue the construction of the event A by adding further steps to the block construction from $x(R)$ until we obtain a target zone $V_{I,J}(x)$ with the property that, for all $u \in V_{0,0}(y)$, $v \in V_{I,J}(x)$ there exists a directed path of $\vec{\mathbb{L}}^3_{\mathrm{alt}}$ from u to v. We let A be the event that the construction successfully attains an r-disk centred in $V_{I,J}(x)$ and then we argue as before. In this case, $\mathbb{P}_p(A) \ge (1 - \epsilon)^{2kJ}$, and we amend the choice of ϵ accordingly. □

Proof of Theorem 2. (a) Let $p_c < \alpha \le \beta \le 1$, $\xi > 0$, and pick $R = R(\alpha, \xi)$ according to Lemma 5.1. Let Δ^γ_u denote the earliest R-disk such that the vertex u is fully γ-connected to Δ^γ_u. Here, 'earliest' means in the ordering induced by the third coordinate value, and, if there is a choice, we take the earliest in some predetermined ordering. Let A^γ_u be the event that such a Δ^γ_u exists. It is elementary, by a 're-start' argument, that

$$\mathbb{P}(A^\gamma_x \mid x \xrightarrow{\gamma} \infty) = 1 \qquad \text{for } \gamma > p_c.$$

By Lemma 5.1,

$$\mathbb{P}\big(\exists z \text{ such that } x \xrightarrow{\alpha} z \xrightarrow{\alpha} \infty,\ y \xrightarrow{\beta} z \mid A^\alpha_x \cap A^\beta_y\big) > 1 - \xi,$$

whence

$$\begin{aligned}\mathbb{P}\big(\exists z \text{ such that } x \xrightarrow{\alpha} z \xrightarrow{\alpha} \infty,\ y \xrightarrow{\beta} z\big) &> (1 - \xi)\mathbb{P}(A^\alpha_x \cap A^\beta_y)\\ &\ge (1 - \xi)\mathbb{P}(x \xrightarrow{\alpha} \infty,\ y \xrightarrow{\beta} \infty).\end{aligned}$$

Since this holds for all $\xi > 0$, the claim of part (a) follows.
(b) The right-continuity of θ is an immediate consequence of the fact that θ is a decreasing limit of continuous non-decreasing functions (cf. [14], p. 203). Since $\theta(p_c) = 0$, by Theorem 1, it follows that θ is continuous on $[0, p_c]$. In order to prove the left-continuity of θ on the interval $(p_c, 1]$, one adapts the argument of [6] in the usual way (see [14], p. 203), making use of the result of part (a). □

6. Proof of Theorem 3

Rather than developing the argument of [16], we make use of a result of [5], where it is proved in the context of the conventional model on $\vec{\mathbb{L}}^3$ that, for sufficiently large p, the undirected graph C is transient almost surely on the event $\{|C| = \infty\}$. The further question is posed in [5] whether this conclusion is valid under the weaker hypothesis that p exceeds the appropriate critical probability, and our Theorem 3 is a positive answer to this question. Although Theorem 3 as stated relates to the alternative model, similar arguments apply for the conventional model.

As explained earlier, we do not include all the details of the required proof, but instead we describe only the salient features. We apologise to those who might have savoured the complicated notation of an overlong full proof, but we hope that readers familiar with [7] will agree with our decision.

We shall construct a block process whose target zones are indexed in the following manner. Let G be the graph having vertex set $\mathbb{Z}^3$ and edge set as follows: for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, we place a directed edge from x to y if and only if $|x_1 - y_1| + |x_2 - y_2| = 1$ and $y_3 - x_3 = 1$. Note that G is a subgraph of $\vec{\mathbb{L}}^3_{\mathrm{alt}}$. The target zones of our block process will be indexed by the set W of vertices of G which are accessible along directed paths from the origin, and the block process will proceed by building connections within regions of $\vec{\mathbb{L}}^3_{\mathrm{alt}}$ represented by the edges joining vertices in W.

Suppose that $p > p_c$. Let $\epsilon > 0$, and choose r, L, K as in (4.1) and Lemma 4.1. We shall assume a bound on ϵ of the form $\epsilon < \epsilon_0$, where the small quantity ϵ_0 will be chosen later. By the usual 're-start argument', if $0 \to \infty$, there exists almost surely an r-disk Δ such that 0 is fully connected to Δ. We use the argument of Section 4 to construct a block process from Δ in which the target zones are indexed by W. This block construction dominates (stochastically) a directed site percolation model on W with some density $1 - \eta(\epsilon)$, where $\eta(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$. Assume for the moment that we may replace the word 'site' with the word 'bond' in the last sentence. It may then be shown, by Proposition 1.2 and Theorem 1.3 of [5] together with some standard arguments concerning electrical networks (see [10, 16]), that C is transient with (conditional) probability at least $1 - \xi(\epsilon)$, for some $\xi(\epsilon)$ satisfying $\xi(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$. Since ϵ was arbitrary, the conclusion will follow. There are two gaps in this argument, namely:

(a) to show that supercritical directed percolation on $\vec{\mathbb{L}}^3_{\mathrm{alt}}$ dominates a certain directed *bond* percolation process on W having large density,
(b) to check that the conclusion of Theorem 1.3 of [5] is valid for the graph G.

An examination of the proof of Theorem 1.3 of [5] reveals that it is easily adapted to the graph G, and one may conclude that there exists a probability measure on directed paths from the origin in G that has exponential intersection tails. We turn therefore to point (a).

We begin with a 'local connection lemma'. Let p, ϵ $(< \epsilon_0)$, r, L, K be as above, and let $S = K + 2r$. Let $B_{l,k} = [-l, l]^2 \times [0, k]$ as before, and let $\mathcal{D}$ be the set of all r-disks centred in $B_{3L,3K}$. For $M \geq 5 \max\{K, L\}$ and $\Delta, \Delta' \in \mathcal{D}$, let $E_{\Delta,\Delta'}$

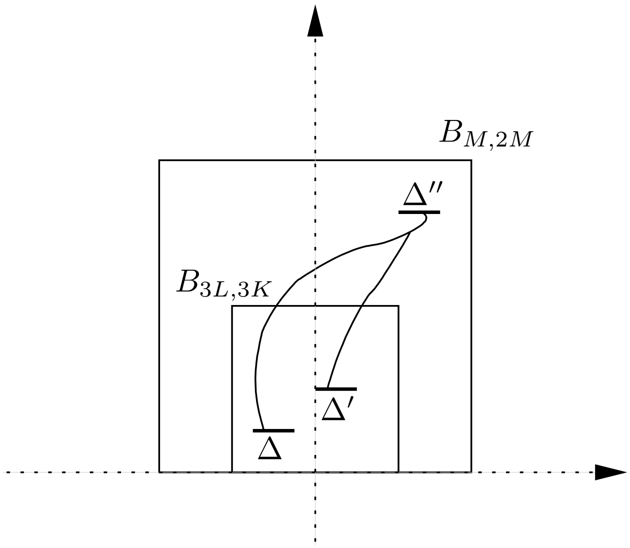

Figure 3. An illustration of the event $E_{\Delta,\Delta'}$.

be the event that there exists an r-disk Δ'' contained entirely in $B_{M,2M}$ such that both Δ and Δ' are fully connected to Δ'' in $B_{M,2M}$. This event is illustrated in Figure 3.

Lemma 6.1. *There exist an integer $M(\epsilon) \geq 5\max\{K, L\}$ and a function $\beta(\epsilon)$ satisfying $\beta(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$ such that*

$$\mathbb{P}_p(E_{\Delta,\Delta'}) \geq 1 - \beta(\epsilon) \quad \textit{for all } \Delta, \Delta' \in \mathcal{D},$$

whenever $M \geq M(\epsilon)$.

Proof. Let $x(r)$ denote the r-disk centred at the vertex x. For $x, y \in \mathbb{Z}^3$, let $F_{x,y}(N)$ be the event that there exists an r-disk Δ'' such that $x(r) \to_{\text{fc}} \Delta''$ and $y(r) \to_{\text{fc}} \Delta''$ in $x + B_{N,N}$. By Lemma 5.1, there exists $\gamma(\epsilon)$ satisfying $\gamma(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$ such that

$$\mathbb{P}_p\Big(\lim_{N\to\infty} F_{x,y}(N)\Big) \geq 1 - \gamma(\epsilon) \quad \text{for all } x, y \in \mathbb{Z}^3.$$

Therefore there exists $N = N_{x(r),y(r)}$ such that

$$\mathbb{P}_p\big(F_{x,y}(N_{x(r),y(r)})\big) \geq 1 - 2\gamma(\epsilon) \quad \text{for all } x, y \in \mathbb{Z}^3.$$

Let $\Delta, \Delta' \in \mathcal{D}$, take $x(r) = \Delta$, $y(r) = \Delta'$, and choose $M = M(\epsilon) \geq 5\max\{K, L\}$ sufficiently large that $\Delta + B_{N_{\Delta,\Delta'},N_{\Delta,\Delta'}} \subseteq B_{M,2M}$ for all $\Delta \in \mathcal{D}$. With this value of M, $\mathbb{P}_p(E_{\Delta,\Delta'}) \geq 1 - 2\gamma(\epsilon)$ uniformly in Δ and Δ'. □

We illustrate next how Lemma 6.1 may be used to show that supercritical directed percolation on $\vec{\mathbb{L}}^3_{\text{alt}}$ dominates a certain *two-dimensional* directed bond percolation process. It is easier to draw pictures in this case, and it will be explained later how to extend the claim to the three-dimensional graph G.

Let $1-\rho(\epsilon)$ be the probability that the two-dimensional block construction of Section 4, initiated from the disk D_r, yields an infinite structure. Since this block process dominates a directed site percolation process having some density $1-\eta(\epsilon)$ where $\eta(\epsilon)\downarrow 0$ as $\epsilon\downarrow 0$, we have by standard arguments (see, for example, [11, 23] and the references in [7]) that there exist $v(\epsilon),\gamma(\epsilon)\geq 0$ such that:

- (a) v is non-increasing in ϵ, and strictly positive when $\rho(\epsilon)<1$,
- (b) $\gamma(\epsilon)\downarrow 0$ as $\epsilon\downarrow 0$,
- (c) for $i,j\in\mathbb{Z}$ such that $j\geq 0$ and $i+j$ is even, if $|i|/j<v(\epsilon)$ then the target zone $V_{i,j}$ is achieved with probability at least $1-\gamma(\epsilon)$ by an open path lying entirely within the convex region

$$\mathcal{C}(\epsilon)=B_{5L,10L}+\left\{(0,x_2,x_3):|x_2|/x_3\leq\frac{L}{2S}\cdot 2v(\epsilon)\right\}.$$

Let $M=M(\epsilon)$ and $\beta(\epsilon)$ be given as in Lemma 6.1. It follows that we may find positive integers I, J depending on ϵ such that:

- (i) IL/M and JS/M are large, say $IL/M, JS/M\geq 10$,
- (ii) $v(\epsilon)<I/J<2v(\epsilon)$,
- (iii) for all r-disks Δ lying in $B_{M,2M}$, there exists with probability at least $1-\gamma(\epsilon)$ an r-disk Δ' centred in $11(0,IL,2JS)+B_{3L,3K}$ such that $\Delta\rightarrow_{\text{fc}}\Delta'$ in $\mathcal{E}(\epsilon)=B_{M,2M}+\mathcal{C}(\epsilon)$.

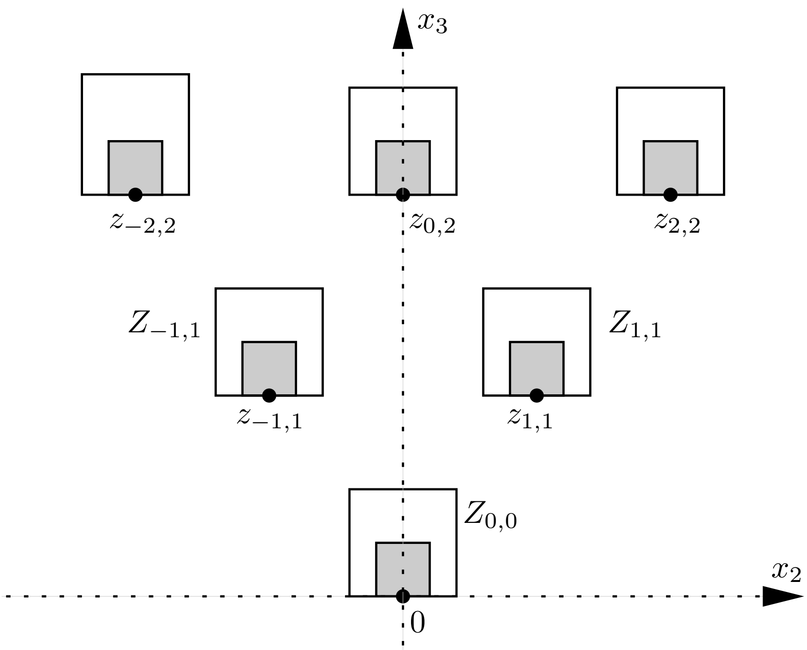

Figure 4. The target zones $Z_{i,j}$. Each smaller box $Z_{i,j}$ is a translate of $B_{3L,3K}$, and is contained in a translate of $B_{M,2M}$.

For integers i, j such that $j \geq 0$ and $i + j$ is even, we define target zones

$$Z_{i,j} = z_{i,j} + B_{3L,3K},$$

where $z_{i,j} = 11(0, iIL, 2jJS)$. See Figure 4.

We now define new block indicator variables $\Theta_{i,j}$ inductively as follows. We set $\Theta_{0,0} = 1$. For $i = \pm 1$, we set $\Theta_{i,1} = 1$ if and only if D_r is fully connected to some r-disk centred in $Z_{i,1}$ by open paths in $\mathcal{E}(\epsilon)$. If this holds, we let $\Delta_{i,1}$ be the earliest such r-disk in $Z_{i,1}$. Having constructed $\{\Theta_{i,j} : j \leq R\}$, we find the $\Theta_{i,R+1}$ as follows. Let $g(i, R+1)$ be the set of all $i' \in \{i-1, i+1\}$ such that $\Theta_{i',R} = 1$. If $g(i, R+1)$ is empty, we set $\Theta_{i,R+1} = 0$. If $g(i, R+1)$ contains a singleton, say the value i', we set $\Theta_{i,R+1} = 1$ if and only if $\Delta_{i',R}$ is fully connected to some r-disk centred in $Z_{i,R+1}$ by open paths lying in $z_{i',R} + \mathcal{E}(\epsilon)$, and we denote by $\Delta_{i,R+1}$ the earliest such r-disk. So far we have retained much of the manner of the construction given in Section 4. However, an important difference arises when $g(i, R+1)$ contains *both* $i-1$ and $i+1$. In this case we set $\Theta_{i,R+1} = 1$ if and only if the following occur:

1. for $i' = i - 1$ and $i' = 1 + 1$, the r-disk $\Delta_{i',R}$ is fully connected to some r-disk centred in $Z_{i,R+1}$ by open paths lying in $z_{i',R} + \mathcal{E}(\epsilon)$,
2. writing $\Delta(i')$ for the earliest such r-disk referred to above, the event $E_{\Delta(i-1),\Delta(i+1)}$ occurs.

If these occur, we denote by $\Delta_{i,R+1}$ the earliest R-disk attained in the definition of $E_{\Delta(i-1),\Delta(i+1)}$. This event is illustrated in Figure 5.

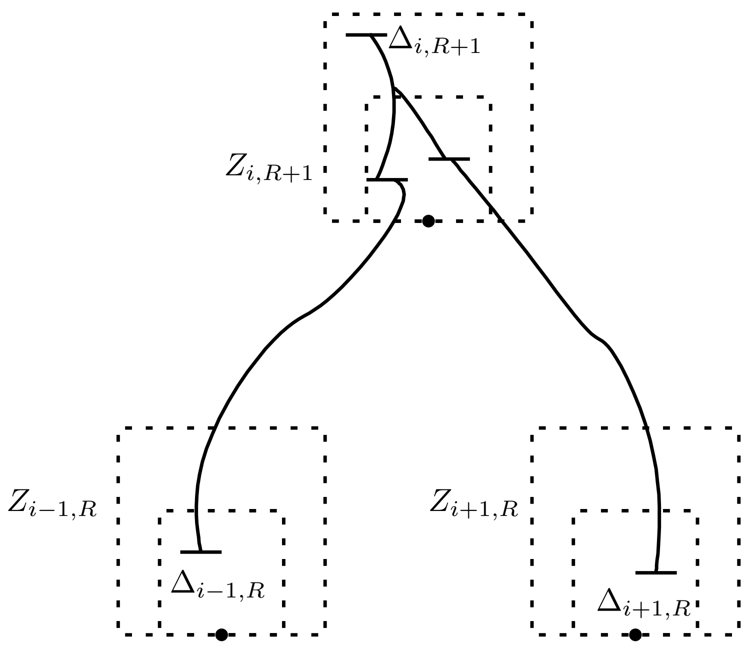

Figure 5. An illustration of the definition of the block variable $\Theta_{i,R+1}$.

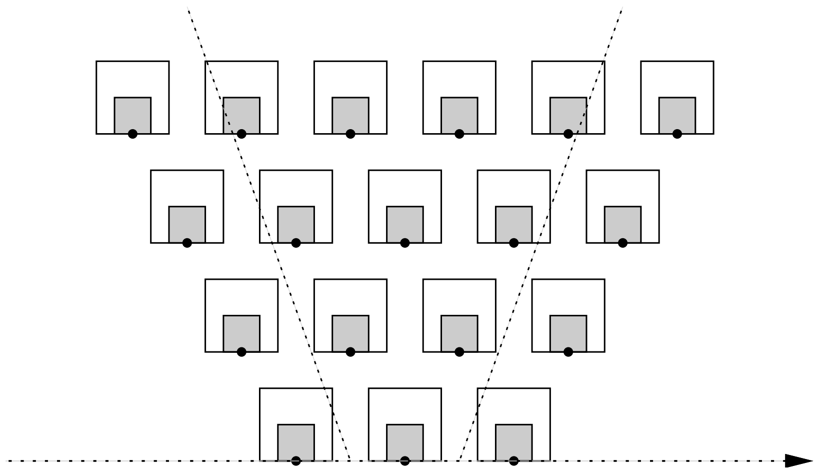

Figure 6. Since we restrict ourselves to open paths lying within a certain 'wedge', the dependence between block variables has a range which is bounded in ϵ ($< \epsilon_0$).

The $\Theta_{i,j}$ are dependent random variables, but the extent of their dependence is limited. Suppose $\{\Theta_{i,j} : j \leq R\}$ have been observed and the corresponding disks $\Delta_{i,j}$ found. The pairs $(\Theta_{i,R+1}, \Delta_{i,R+1})$, $-R-1 \leq i \leq R+1$, have some interdependence owing to the fact that the open paths from different $\Delta_{i,R}$ may lie close to one another. Using conditions (i)–(iii) together with the fact that $v(\epsilon)$ is decreasing in ϵ, there exists an integer $T = T(\epsilon_0)$ such that: conditional on the pairs $(\Theta_{i,R}, \Delta_{i,R})$, $-R \leq i \leq R$, the family of pairs $(\Theta_{i,R+1}, \Delta_{i,R+1})$, $-R-1 \leq i \leq R+1$, is T-dependent (see [14], page 178 for a definition of T-dependence). This observation is illustrated in Figure 6.

Each step in the above construction of the $\Theta_{i,j}$ is successful with probability at least $1 - 2\gamma(\epsilon) - \beta(\epsilon)$, which approaches 1 as $\epsilon \downarrow 0$. Since T is an absolute constant, we deduce by the comparison theorem of [25] (see also Theorem (7.65) of [14]), that the $\{\Theta_{i,j}\}$, together with the successful connections between the r-disks $\Delta_{i,j}$, dominate (stochastically) the open cluster at the origin of a directed *bond* percolation process on $\vec{\mathbb{L}}^2$ having density approaching 1 as $\epsilon \downarrow 0$.

Several details are missing from the foregoing argument, of which one is an account of the 'steering' necessary to achieve property (iii) above. This follows a standard route, and is omitted.

We return finally to point (a) before Lemma 6.1. It is required to show that the two-dimensional construction of the $\Theta_{i,j}$ may be extended to a three-dimensional construction with target zones indexed by the set W. This we achieve with the aid of some pictures. The target zones are now

$$Z_{i,j,k} = z_{i,j,k} + B_{3L,3K},$$

where $z_{i,j,k} = 11(iIL, jIL, 4kJS)$, as (i,j,k) ranges over integer vectors satisfying $|i| \leq k$, $|j| \leq k$, and $i+k$ and $j+k$ are even. A plan of these target zones is drawn in Figure 7. Boxes of the form $11(iIL, jIL, 2kJS) + B_{3L,3K}$ for odd values of k are termed 'intermediate zones'.

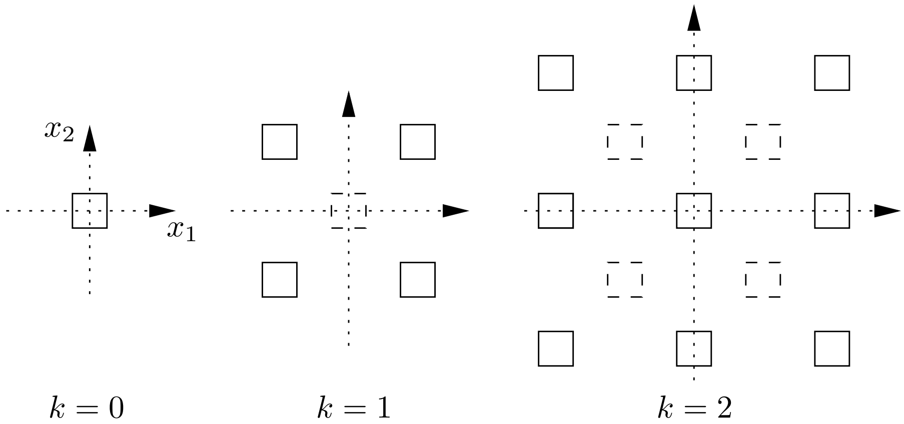

Figure 7. A plan of the target zones $Z_{i,j,k}$.

We describe next the open connections sought in defining the block indicator variables $\Theta_{i,j,k}$. First we set $\Theta_{0,0,0} = 1$. For $i, j \in \{-1, 1\}$, we declare $\Theta_{i,j,1} = 1$ if and only if:

(a) there exists an r-disk Δ centred in the intermediate zone $11(iL, 0, 2JS) + B_{3L,3K}$ such that $D_r \to_{\rm fc} \Delta$ in a certain convex region $\mathcal{E}$, and
(b) there exists an r-disk Δ' in $Z_{i,j,1}$ such that $\Delta \to_{\rm fc} \Delta'$ within a certain translate/rotation of $\mathcal{E}$.

The region $\mathcal{E}$ corresponds to the $\mathcal{E}(\epsilon)$ used above. When such connections exist, we let $\Delta_{i,j,1}$ be the earliest disk in $Z_{i,j,1}$ which is thus reached from D_r.

The next step is similar to that described in the two-dimensional case. We omit most of the details, but concentrate on one illustrative example. Suppose for the sake of the illustration that $\Theta_{i,j,1} = 1$ for all i, j, and that we are seeking a definition of $\Theta_{0,0,2}$. We set $\Theta_{0,0,2} = 1$ if and only if the following hold:

(i) there exists an r-disk $\Delta(-1, 1)$ (respectively $\Delta(1, 1)$) centred in the intermediate zone $11(0, IL, 6JS) + B_{3L,3K}$ such that $\Delta_{-1,1,1} \to_{\rm fc} \Delta(-1, 1)$ (respectively $\Delta_{1,1,1} \to_{\rm fc} \Delta(1, 1)$) by paths lying within a certain translate/rotation of $\mathcal{E}$,
(ii) there exists an r-disk Δ_1 such that $\Delta(-1, 1) \to_{\rm fc} \Delta_1$ and $\Delta(1, 1) \to_{\rm fc} \Delta_1$ in $11(0, IL, 6JS) + B_{M,2M}$,
(iii) there exists an r-disk $\Delta(-1, -1)$ (respectively $\Delta(1, -1)$) centred in the intermediate zone $11(0, -IL, 6JS)+B_{3L,3K}$ such that $\Delta_{-1,-1,1} \to_{\rm fc} \Delta(-1, -1)$ (respectively $\Delta_{1,-1,1} \to_{\rm fc} \Delta(1, -1)$) by paths lying within a certain translate/rotation of $\mathcal{E}$,
(iv) there exists an r-disk Δ_{-1} such that $\Delta(-1, -1) \to_{\rm fc} \Delta_{-1}$ and $\Delta(1, -1) \to_{\rm fc} \Delta_{-1}$ in $11(0, -IL, 6JS) + B_{M,2M}$,
(v) there exists an r-disk Δ'_1 (respectively Δ'_{-1}) centred in the box $z_{0,0,2} + B_{3L,3K}$ such that $\Delta_1 \to_{\rm fc} \Delta'_1$ (respectively $\Delta_{-1} \to_{\rm fc} \Delta'_{-1}$) by paths lying within certain translates/rotations of $\mathcal{E}$,
(vi) there exists an r-disk Δ in $Z_{0,0,2}$ such that $\Delta'_1 \to_{\rm fc} \Delta$ and $\Delta'_{-1} \to_{\rm fc} \Delta$ in $Z_{0,0,2}$.

If these events occur, we define $\Theta_{0,0,2} = 1$, and we let $\Delta_{0,0,2}$ be the earliest r-disk Δ thus accessible in (v).

One may define the variables $\Theta_{i,j,k}$ in a similar inductive way. As before, it may be shown via the stochastic domination theorem of [25] that the block process dominates a bond percolation process with high density, and our sketch of the proof of Theorem 3 is complete.

7. Infinite-range percolation

We consider next a long-range directed percolation model which extends our earlier results for the alternative lattice $\vec{\mathbb{L}}^d_{\text{alt}}$. Let $d \geq 2$, and let $\mathbf{p} = \{p(\mathbf{x})\colon \mathbf{x} \in \mathbb{Z}^{d-1}\}$ be a collection of numbers satisfying $0 \leq p(\mathbf{x}) < 1$. Consider the vertex set $\mathbb{Z}^d$, and write $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$ as $x = (\mathbf{x}, t)$ where $\mathbf{x} = (x_1, x_2, \ldots, x_{d-1}) \in \mathbb{Z}^{d-1}$ and $t = x_d \in \mathbb{Z}$. For every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{d-1}$ and $t \in \mathbb{Z}$, we place a directed edge from $(\mathbf{x}, t)$ to $(\mathbf{y}, t+1)$ with probability $p(\mathbf{y} - \mathbf{x})$. Each such pair is joined by an edge independently of the presence or absence of other edges.

We shall use the same notation as earlier. For example, we write $x \to y$ if there exists a directed path from x to y, and we let $\theta(\mathbf{p}) = \mathbb{P}_{\mathbf{p}}(0 \to \infty)$, where $\mathbb{P}_{\mathbf{p}}$ denotes the appropriate probability measure.

We shall require a certain amount of symmetry, and to this end we shall assume that:

(a) $\mathbf{p}$ is invariant under sign changes of components, in that $p(\mathbf{x}) = p(\mathbf{x}')$ whenever $\mathbf{x}'$ is obtained from $\mathbf{x}$ by changing the signs of any of the $d-1$ components of $\mathbf{x}$, and
(b) $\mathbf{p}$ is invariant under permutations of components, in that $p(\mathbf{x}) = p(\pi\mathbf{x})$ where π is a permutation of $1, 2, \ldots, d-1$, and $\pi(x_1, x_2, \ldots, x_{d-1}) = (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(d-1)})$.

The *range* R of the process is defined as $R = \sup\{|\mathbf{x}| : p(\mathbf{x}) > 0\}$, and the process is said to have *infinite range* if $R = \infty$.

Under what further assumptions on $\mathbf{p}$ may one adapt the ideas underlying the block construction of Section 4? We will present sufficient conditions on $\mathbf{p}$, and it will follow in the usual way that, when these conditions are valid, the usual conclusions follow, including the continuity of slab critical points, the (suitably generalized) uniqueness result of Theorem 2(a), the fact that the *critical* process (for a suitable parametric family of functions $\mathbf{p}$ satsfying a further condition of continuity) dies out, and many other observations. The details of such consequences are not included here, since they follow already familiar lines. Instead, we make specific our sufficient conditions on $\mathbf{p}$, and we outline the steps which follow for the required block construction.

Infinite-range *undirected* percolation has been studied in one and higher dimensions in [2, 15], and a block construction was developed in [26] subject to a rather severe condition on the decay rate of probabilities of long-range edges. Infinite-range *directed* percolation, and particularly some of the facts referred to above, are relevant to the scaling limit of critical directed percolation in high dimensions, proved in [21, 29]. The decay rate required by the forthcoming conditions on the function $\mathbf{p}$ is substantially weaker than that assumed in [26] for undirected percolation.

For $\mathbf{x} = (x_1, x_2, \ldots, x_{d-1}) \in \mathbb{Z}^{d-1}$, we define $|\mathbf{x}|_\infty = \max\{|x_i| : 1 \le i \le d-1\}$, and we write

$$\Sigma_u = \sum_{\mathbf{x}:|\mathbf{x}|_\infty > u} p(\mathbf{x}) \quad \text{for } u > 0.$$

The relevant conditions on $\mathbf{p}$ are the following.

I. *Summability.* $\sum_{\mathbf{x}\in\mathbb{Z}^{d-1}} p(\mathbf{x}) < \infty$.

II. *Aperiodicity.* For every $\mathbf{x} \in \mathbb{Z}^{d-1}$, the set $\{t : \mathbb{P}_{\mathbf{p}}(0 \to (\mathbf{x}, t)) > 0\}$ has greatest common divisor 1.

III. *Tail regularity.* There exists an integer $\alpha > 1$ and a real $\xi \in (0, 1)$ such that $\Sigma_{\alpha h} \le \xi \Sigma_h$ for all $h \in \mathbb{R}$ satisfying $h \ge 1$.

We next discuss these conditions. Condition I holds if and only if every vertex has almost surely finite vertex degree. Condition II is a convenience but is not essential. If Condition II fails, then one would sometimes need to restrict oneself to an appropriate subset of $\mathbb{Z}^d$. Condition III is a condition of smoothness on the manner in which $p(\mathbf{x})$ decays for large $|\mathbf{x}|_\infty$, and this will be required for the renormalization argument.

Condition III is not over severe. Assume for the sake of illustration that $p(\mathbf{x}) = g(|\mathbf{x}|_\infty)$ for large $|\mathbf{x}|_\infty$, where $g(v) = v^{-\beta}$. Then Condition III is satisfied whenever Condition I holds, namely if $\beta > d-1$. The condition is however not satisfied if, for example, $g(v) = v^{-d+1}(\log v)^{-2}$.

It is easy to see, by the symmetry of $\mathbf{p}$, that Condition III implies

$$\sum_{\substack{\mathbf{x}:|\mathbf{x}|_\infty \le \alpha h \\ x_1 > h}} p(\mathbf{x}) \ge \frac{\Sigma_h - \Sigma_{\alpha h}}{2(d-1)} \ge \frac{(1-\xi)\Sigma_h}{2(d-1)} \ge \frac{1-\xi}{2(d-1)} \sum_{\mathbf{x}:x_1>h} p(\mathbf{x}), \tag{7.1}$$

and we shall make use of this fact later.

Let us assume henceforth that $\mathbf{p}$ satisfies Conditions I, II, III for some pair α, ξ, and we assume as before that $d = 3$. We claim that the 'usual theorems' follow, and to this end we now sketch the necessary extra steps in order to achieve a block construction as in Section 4. The principal step is to establish an equivalent of Lemma 4.1. Whereas Lemma 4.1 concerned the numbers of points on the top and sides of a block $B_{L,K}$ which are endpoints of directed paths of the block originating from the disk D_r, in the infinite-range setting we concentrate on the number $N_{\mathrm{t}}(L, K)$ of such points on the top of $B_{L,K}$, and the mean number $\widetilde{N}_{\mathrm{s}}(L, K)$ of edges exiting $B_{L,K}$ by its sides. That is, we replace $N_{\mathrm{s}}(L, K)$ by

$$\widetilde{N}_{\mathrm{s}}(L, K) = \sum_{\substack{x=(\mathbf{x},t) \\ y=(\mathbf{y},t+1) \\ 0\le t<K}} p(\mathbf{y} - \mathbf{x}) \tag{7.2}$$

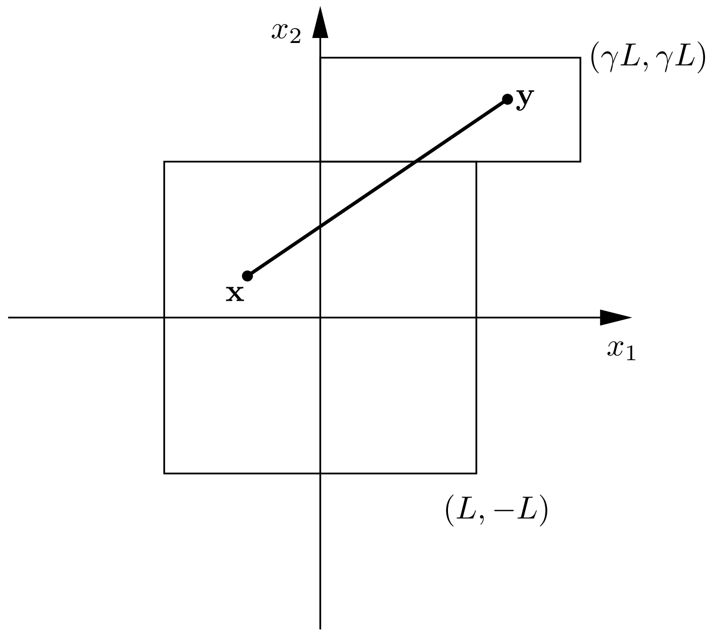

Figure 8. A map of (x_1, x_2)-space. The set of possible values of $\mathbf{y}$ in (7.3) is in the upper rectangle.

where the summation is over $x \in B_{L,K}$ such that $D_r \to x$ in $B_{L,K}$, and over $y \in \{[-\gamma L, \gamma L]^2 \setminus [-L, L]^2\} \times [1, K]$, where $\gamma = 2\alpha - 1$. For $\mathbf{v} \in \{-1, +1\}^3$, we consider the sub-facet of the sides of $B_{L,K}$ indexed by $\mathbf{v}$, and we define $\widetilde{N}_{\mathrm{s}}^{\mathbf{v}}(L, K)$ as follows. For simplicity, assume this sub-facet is $[0, L] \times \{L\} \times [0, K]$, and let

$$\widetilde{N}_{\mathrm{s}}^{\mathbf{v}}(L, K) = \sum_{\substack{x=(\mathbf{x},t) \\ y=(\mathbf{y},t+1) \\ 0 \le t < K}}^{\mathbf{v}} p(\mathbf{y} - \mathbf{x}) \tag{7.3}$$

where the summation is as in (7.2) but with y restricted to the region $[0, \gamma L] \times (L, \gamma L] \times [0, K]$. See Figure 8.

With a few minor changes, the proof of Lemma 4.1 may be adapted to obtain the corresponding conclusion with $N_{\mathrm{s}}^{\mathbf{v}}$ replaced by $\widetilde{N}_{\mathrm{s}}^{\mathbf{v}}$. We highlight one difference, namely that related to Condition III. For $\mathbf{x} \in [-L, L]^2$, we set

$$\nu_L(\mathbf{x}) = \sum_{\mathbf{y} \notin [-L, L]^2} p(\mathbf{y} - \mathbf{x}), \tag{7.4}$$

and

$$Q = \sum_{\substack{x=(\mathbf{x},t) \\ 0 \le t < K}} \nu_L(\mathbf{x}) = \sum_{\substack{x=(\mathbf{x},t) \\ y=(\mathbf{y},t+1) \\ 0 \le t < K}} p(\mathbf{y} - \mathbf{x}) \tag{7.5}$$

where the sums are over $x \in B_{L,K}$ such that $D_r \to x$ in $B_{L,K}$, and over $y \in \{(-\infty,\infty)^2 \setminus [-L,L]^2\} \times [1,K]$. We require a statement of the form: $\widetilde{N}_{\mathrm{s}}(L,K)$ is large if and only if Q is large.

Lemma 7.1. *We have that* $\frac{1}{8}(1-\xi)Q \le \widetilde{N}_{\mathrm{s}}(L,K) \le Q$.

Proof. For $\mathbf{x} \in [-L,L]^2$, let

$$\nu_L^\gamma(\mathbf{x}) = \sum_{\mathbf{y}\in[-\gamma L,\gamma L]^2 \setminus [-L,L]^2} p(\mathbf{y}-\mathbf{x}). \tag{7.6}$$

For $i = 1,2$ and $\eta = \pm$, let $\nu_L^{i,\eta}(\mathbf{x})$ (respectively, $\nu_L^{\gamma,i,\eta}(\mathbf{x})$) be given as in (7.4) (respectively, (7.6)) but with the restriction to vertices y satisfying $\eta y_i > L$. Evidently, by (7.1),

$$\nu_L^\gamma(\mathbf{x}) \ge \tfrac{1}{2}\sum_{i,\eta}\nu_L^{\gamma,i,\eta}(\mathbf{x}) \ge \tfrac{1}{8}(1-\xi)\sum_{i,\eta}\nu_L^{i,\eta}(\mathbf{x}) \ge \tfrac{1}{8}(1-\xi)\nu_L(\mathbf{x}).$$

Therefore,

$$\widetilde{N}_{\mathrm{s}}(L,K) = \sum_{\substack{x=(\mathbf{x},t)\\ 0\le t<K}} \nu_L^\gamma(\mathbf{x}) \ge \sum_{\substack{x=(\mathbf{x},t)\\ 0\le t<K}} \tfrac{1}{8}(1-\xi)\nu_L(\mathbf{x}) = \tfrac{1}{8}(1-\xi)Q,$$

as required, where the sums are over $x \in B_{L,K}$ such that $D_r \to x$ in $B_{L,K}$. The other inequality of the lemma is a triviality. □

Having achieved an infinite-range version of Lemma 4.1, one embarks on the block construction, and it is here that one sees the value of the definition of the $\widetilde{N}_{\mathrm{s}}^{\mathbf{v}}(L,K)$. The maximum lateral extension of the points y in (7.2) is γ times the dimension of the sides of $B_{L,K}$, where γ is a constant which is independent of L and K. It is the last fact which enables a block construction in which each step uses a number l of extensions of the basic step of Lemma 4.1, the number l being independent of L and K. With some care but with no further ideas of consequence, it follows that, with an appropriate choice of parameters, the block process stochastically dominates a conventional directed site percolation process having density close to 1.

We finish with a discussion of the infinite-range analogue of the statement $\theta(p_{\mathrm{c}}) = 0$. Let $\{\mathbf{p}^r : 0 \le r \le 1\}$ be a parametric family of functions satisfying conditions (a) and (b) at the beginning of this section, and suppose that every $\mathbf{p}^r$ satisfies conditions I, II, III 'uniformly in r', in the sense that, for some M:

I′. $\sum_{\mathbf{x}} p^r(\mathbf{x}) \le M < \infty$ for all r,

II′. for every $\mathbf{x} \in \mathbb{Z}^{d-1}$, the set $\{t : \mathbb{P}_{\mathbf{p}^r}(0 \to (\mathbf{x},t)) > 0\}$ is independent of r, and has greatest common divisor 1,

III′. there exist an integer α and a real $\xi \in (0,1)$ such that

$$\Sigma_u^r = \sum_{\mathbf{x}:|\mathbf{x}|_\infty > u} p^r(\mathbf{x})$$

satisfies $\Sigma_{\alpha h}^r \le \xi \Sigma_h^r$ for all $h \ge 1$.

We impose a further condition, namely:

IV′. for every $\mathbf{x} \in \mathbb{Z}^{d-1}$, the quantity $p^r(\mathbf{x})$ is a continuous and strictly increasing function of r.

Suppose that

$$r_c = \sup\{r : \mathbb{P}_{\mathbf{p}^r}(0 \to \infty) = 0\}$$

satisfies $0 < r_c < 1$. It follows under Conditions I′–IV′ by the block construction and the standard argument quoted in Section 4 that $\theta(\mathbf{p}^{r_c}) = 0$.

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