COUNTING SELF-AVOIDING WALKS

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Abstract. The *connective constant* $\mu(G)$ of a graph G is the asymptotic growth rate of the number of self-avoiding walks on G from a given starting vertex. We survey three aspects of the dependence of the connective constant on the underlying graph G. Firstly, when G is cubic, we study the effect on $\mu(G)$ of the Fisher transformation (that is, the replacement of vertices by triangles). Secondly, we discuss upper and lower bounds for $\mu(G)$ when G is regular. Thirdly, we present strict inequalities for the connective constants $\mu(G)$ of vertex-transitive graphs G, as G varies. As a consequence of the last, the connective constant of a Cayley graph of a finitely generated group decreases strictly when a new relator is added, and increases strictly when a non-trivial group element is declared to be a generator. Special prominence is given to open problems.

1. Introduction

A *self-avoiding walk* (abbreviated to SAW) on a graph $G = (V, E)$ is a path that visits no vertex more than once. An example of a SAW on the square lattice is drawn in Figure 1.1. SAWs were first introduced in the chemical theory of polymerization (see Flory [10]), and their critical behaviour has attracted the abundant attention since of mathematicians and physicists (see, for example, the book of Madras and Slade [32] or the lecture notes [4]).

Let $\sigma_n(v)$ be the number of n-step SAWs on G starting at the vertex v. The following fundamental theorem of Hammersley asserts the existence of an asymptotic growth rate for $\sigma_n(v)$ as $n \to \infty$. (See the start of Section 2 for a definition of (quasi-)transitivity.)

Theorem 1.1. [22] *Let $G = (V, E)$ be an infinite, connected, quasi-transitive graph with finite vertex-degrees. There exists $\mu = \mu(G) \in$*

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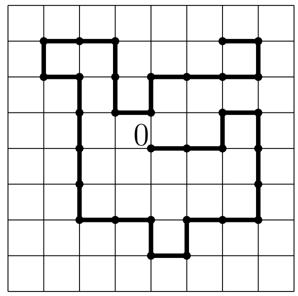

FIGURE 1.1. A SAW from the origin of the square lattice.

$[1, \infty)$, *called the* connective constant *of* G, *such that*

$$\lim_{n\to\infty} \sigma_n(v)^{1/n} = \mu, \qquad v \in V. \tag{1.1}$$

We review here recent work from [16, 17, 19] on the dependence of $\mu(G)$ on the choice of graph G.

For what graphs G is $\mu(G)$ known exactly? There are a number of such graphs, which should be regarded as atypical in this regard. We mention the *ladder* $\mathbb{L}$, the hexagonal lattice $\mathbb{H}$, and the *bridge graph* $\mathbb{B}_\Delta$ with degree $\Delta \geq 2$ of Figure 1.2, for which

$$\mu(\mathbb{L}) = \tfrac{1}{2}(1+\sqrt{5}), \quad \mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}, \quad \mu(\mathbb{B}_\Delta) = \sqrt{\Delta - 1}. \tag{1.2}$$

See [2, p. 184] and [8] for the first two calculations.

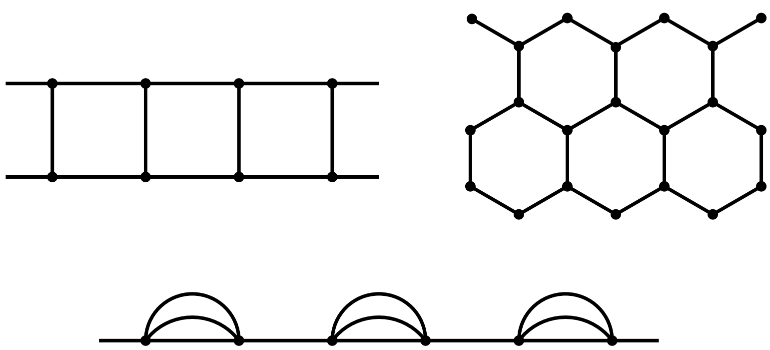

FIGURE 1.2. Three regular graphs: the ladder graph $\mathbb{L}$, the hexagonal tiling $\mathbb{H}$ of the plane, and the bridge graph $\mathbb{B}_\Delta$ (with $\Delta = 4$) obtained from $\mathbb{Z}$ by joining every alternate pair of consecutive vertices by $\Delta - 1$ parallel edges.

In contrast, the connective constant of the square grid $\mathbb{Z}^2$ is unknown, and a substantial amount of work has been devoted to obtaining good bounds. The best rigorous bounds known currently to the authors are those of [25, 35], namely (to 5 significant figures)

$$2.6256 \leq \mu(\mathbb{Z}^2) \leq 2.6792,$$

and more precise numerical estimates are available, including the estimate $\mu \approx 2.63815\ldots$ of [24].

We make some remarks about the three graphs of Figure 1.2. There is a correspondence between the Fibonacci sequence and counts of SAWs on the ladder graph $\mathbb{L}$, whereby one obtains that $\mu(\mathbb{L})$ equals the golden ratio $\phi := \frac{1}{2}(1+\sqrt{5})$. We ask in Section 3 (of the current article) whether $\mu(G) \geq \phi$ for all simple, cubic, vertex-transitive graphs. Amongst a certain category of Δ-regular graphs permitted to possess multiple edges, the bridge graph $\mathbb{B}_\Delta$ is extremal in the sense that $\mu(\mathbb{B}_\Delta)$ is smallest. See the discussion of Section 3.

The proof that $\mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}$ by Duminil-Copin and Smirnov [8] is a very significant recent result. The value $\sqrt{2+\sqrt{2}}$ emerged in the physics literature through work of Nienhuis [34] motivated originally by renormalization group theory. Its proof in [8] is based on the construction of an observable with some properties of discrete holomorphicity, complemented by a neat use of the bridge decomposition introduced by Hammersley and Welsh [23].

It is a beautiful open problem to prove that a random n-step SAW from the origin of $\mathbb{Z}^2$ converges, when suitably re-scaled, to the Schramm–Loewner curve $\mathrm{SLE}_{8/3}$. This important conjecture has been discussed and formalized by Lawler, Schramm, and Werner [28].

Question 1. *Does a uniformly distributed n-step SAW on $\mathbb{Z}^2$ converge, when suitably rescaled, to the random curve $\mathrm{SLE}_{8/3}$?*

There is an important class of results usually referred to as the 'pattern theorem'. In Kesten's original paper [27] devoted to $\mathbb{Z}^2$, a *proper internal pattern* $\mathcal{P}$ is defined as a finite SAW with the property that, for any $k \geq 1$, there exists a SAW containing at least k translates of $\mathcal{P}$. The pattern theorem states that: for a given proper internal pattern $\mathcal{P}$, there exists $a > 0$ such that the number of n-step SAWs from the origin 0, containing no more than an translates of $\mathcal{P}$, is exponentially smaller than the total $\sigma_n := \sigma_n(0)$.

The pattern theorem may be used to prove for this bipartite graph that

$$\lim_{n\to\infty} \frac{\sigma_{n+2}}{\sigma_n} = \mu^2.$$

The following stronger statement has been open since Kesten's paper [27], see the discussion at [32, p. 244].

Question 2. *Is it the case for SAWs on $\mathbb{Z}^2$ that $\sigma_{n+1}/\sigma_n \to \mu$?*

Previous work on SAWs tends to have been focussed on specific graphs such as the cubic lattices $\mathbb{Z}^d$ and certain two-dimensional lattices. In contrast, the results of [16, 17, 19], surveyed here, are directed at large classes of regular graphs, often but not exclusively *transitive* graphs. The work reviewed here may be the first systematic study of SAWs on general vertex-transitive and quasi-transitive graphs.

In Section 2, we describe the Fisher transformation and its effect on counts of SAWs for cubic (or semi-cubic) graphs. Universal bounds for the connective constant $\mu(G)$ of Δ-regular graphs G are presented in Section 3. Strict inequalities for $\mu(G)$ are presented in Section 4. The question addressed there is the following: if G is a strict subgraph of G', under what conditions on the pair (G, G') is it the case that $\mu(G) < \mu(G')$? Two sufficient conditions of an algebraic nature are presented, with applications (in Section 5) to Cayley graphs of finitely generated groups.

A number of questions are included in this review. The inclusion of a question does not of itself imply either difficulty or importance.

Note added at revision. Since this review was written in 2013, the authors have continued their project with papers [18, 20], which are directed at the question of 'locality' of connective constants: to what degree is the value of the connective constant of a vertex-transitive graph determined by knowledge of a large ball centred at a given vertex?

2. The Fisher transformation and the golden mean

This section is devoted to a summary of the effect on $\mu(G)$ of the so-called *Fisher transformation*. We begin with a discussion of transitivity.

The automorphism group of the graph $G = (V, E)$ is denoted $\mathrm{Aut}(G)$, and the identity automorphism is written ι. A subgroup $\Gamma \subseteq \mathrm{Aut}(G)$ is said to *act transitively* on G if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It is said to *act quasi-transitively* if there exists a finite set W of vertices such that, for $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called *vertex-transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively) on G.

An automorphism γ is said to *fix* a vertex v if $\gamma v = v$. The subgroup Γ is said to *act freely* on G (or on the vertex-set V) if: whenever there exist $\gamma \in \Gamma$ and $v \in V$ with $\gamma v = v$, then $\gamma = \iota$.

Let v be a vertex with degree 3. The *Fisher transformation* acts at v by replacing it by a triangle, as illustrated in Figure 2.1. The Fisher

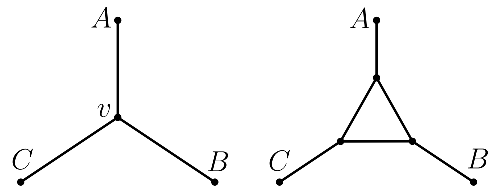

FIGURE 2.1. The Fisher triangulation of the star.

transformation has been valuable in the study of the relations between Ising, dimer, and general vertex models (see [7, 9, 29, 30]), and also in the calculation of the connective constant of the Archimedean lattice $(3, 12^2)$ (see, for example, [14, 21, 26]). The Fisher transformation may be applied at every vertex of a cubic graph (that is, a graph with every vertex of degree 3), of which the hexagonal and square/octagon lattices are examples. We describe next the Fisher transformation in the context of self-avoiding walks.

A graph G is called *simple* if it has no multiple edges. Assume that $G = (V, E)$ is quasi-transitive, connected, and simple. By Theorem 1.1, G has a well-defined connective constant $\mu = \mu(G)$ satisfying (1.1). Suppose, in addition, that G is cubic, and write $F(G)$ for the graph obtained by applying the Fisher transformation at every vertex. The automorphism group of G induces an automorphism subgroup of $F(G)$, so that $F(G)$ is quasi-transitive and has a well-defined connective constant. It is noted in [17], and probably elsewhere also, that the connective constants of G and $F(G)$ have a simple relationship. This conclusion, and its iteration, are given in the next theorem, in which $\phi := \frac{1}{2}(1 + \sqrt{5})$ denotes the golden mean.

Theorem 2.1. [17, Thm 3.1] *Let G be an infinite, quasi-transitive, connected, cubic graph, and consider the sequence $(G_k : k = 0, 1, 2, \dots)$ given by $G_0 = G$ and $G_{k+1} = F(G_k)$.*

(a) *The connective constants $\mu_k := \mu(G_k)$ satisfy $\mu_k^{-1} = g(\mu_{k+1}^{-1})$ where $g(x) = x^2 + x^3$.*
(b) *The sequence μ_k converges monotonely to ϕ, and*

$$-\left(\tfrac{4}{7}\right)^k \le \mu_k^{-1} - \phi^{-1} \le \left[\tfrac{1}{2}(7 - \sqrt{5})\right]^{-k}, \qquad k \ge 1.$$

The idea underlying part (a) is that, at each vertex v visited by a SAW π on G_k, one may replace that vertex by either of the two paths around the 'Fisher triangle' at v. Some book-keeping is necessary with this argument, and this is best done via generating functions (2.1).

We turn now to the Fisher transformation in the context of a 'semi-cubic' graph. A graph is called *bipartite* if its vertices can be coloured black or white in such a way that every edge links a black vertex and a white vertex.

Theorem 2.2. [17, Thm 3.3] *Let G be an infinite, connected, bipartite graph with vertex-sets coloured black and white, and suppose the coloured graph is quasi-transitive, and every black vertex has degree 3. Let $\widetilde{G}$ be the graph obtained by applying the Fisher transformation at each black vertex. The connective constants μ and $\widetilde{\mu}$ of G and $\widetilde{G}$, respectively, satisfy $\mu^{-2} = h(\widetilde{\mu}^{-1})$, where $h(x) = x^3 + x^4$.*

Example 2.3. *Theorem 2.2 implies an exact value of a connective constant that does not appear to have been noted previously. Take $G = \mathbb{H}$, the hexagonal lattice with connective constant $\mu = \sqrt{2+\sqrt{2}} \approx 1.84776$, see [8]. The ensuing lattice $\widetilde{\mathbb{H}}$ is illustrated in Figure 2.2, and its connective constant $\widetilde{\mu}$ satisfies $\mu^{-2} = h(\widetilde{\mu}^{-1})$, which may be solved to obtain $\widetilde{\mu} \approx 1.75056$.*

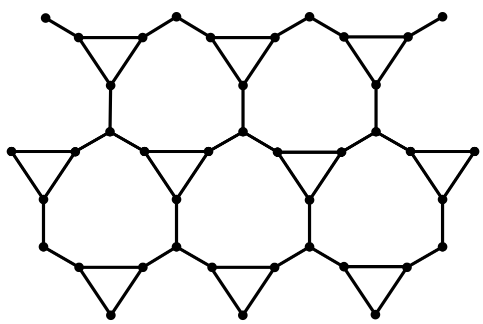

FIGURE 2.2. The lattice $\widetilde{\mathbb{H}}$ derived from the hexagonal lattice $\mathbb{H}$ by applying the Fisher transformation at alternate vertices. Its connective constant $\widetilde{\mu}$ is the root of the equation $x^{-3} + x^{-4} = 1/(2+\sqrt{2})$.

The proofs of Theorems 2.1–2.2 are based on the generating function $Z_v(x)$ of SAWs defined by

$$Z_v(x) = \sum_{w \in \Sigma(v)} x^{|w|}, \tag{2.1}$$

where $\Sigma(v)$ is the set of all finite SAWs starting from a given vertex v, and $|w|$ is the length of w. Viewed as a power series, $Z_v(x)$ has radius of convergence $1/\mu$.

The connective constant is altered by application of the Fisher transformation as described in Theorem 2.1(a). Critical exponents, on the other hand, have values that are not altered. The reader is referred to [4, 32] and the references therein for general accounts of critical exponents for SAWs. The three exponents that have received most attention in the study of SAWs are as follows.

We consider only the case of SAWs in finite-dimensional spaces, thus excluding, for example, the hyperbolic space of [33]. Suppose for concreteness that there exists a periodic, locally finite embedding of G into $\mathbb{R}^d$ with $d \geq 2$, and no such embedding into $\mathbb{R}^{d-1}$. The case of general G has not been studied extensively, and most attention has been paid to the hypercubic lattice $\mathbb{Z}^d$.

The exponent γ. It is believed (when $d \neq 4$) that the generic behaviour of $\sigma_n(v)$ is given by:

$$\sigma_n(v) \sim A_v n^{\gamma-1}\mu^n, \qquad \text{as } n \to \infty,\ v \in V, \tag{2.2}$$

for constants $A_v > 0$ and $\gamma \in \mathbb{R}$. The value of the 'critical exponent' γ is believed to depend on d only, and not further on the choice of graph G. Furthermore, it is believed (and largely proved, see the account in [32]) that $\gamma = 1$ when $d \geq 4$. In the borderline case $d = 4$, (2.2) should hold with $\gamma = 1$ and subject to the correction factor $(\log n)^{1/4}$.

The exponent η. Let $v, w \in V$, and

$$Z_{v,w}(x) = \sum_{n=0}^{\infty} \sigma_n(v,w)x^k, \qquad x > 0,$$

where $\sigma_n(v,w)$ is the number of n-step SAWs with endpoints v, w. It is known under certain circumstances that the generating functions $Z_{v,w}$ have radius of convergence μ^{-1} (see [32, Cor. 3.2.6]), and it is believed that there exists an exponent η and constants $A'_v > 0$ such that

$$Z_{v,w}(\mu^{-1}) \sim A'_v d_G(v,w)^{-(d-2+\eta)}, \qquad \text{as } d_G(v,w) \to \infty,$$

where $d_G(v,w)$ is the graph-distance between v and w. Furthermore, η satisfies $\eta = 0$ when $d \geq 4$.

The exponent ν. Let $\Sigma_n(v)$ be the set of n-step SAWs from v, and write $\langle\cdot\rangle_n^v$ for expectation with respect to the uniform measure on $\Sigma_n(v)$. Let $\|\pi\|$ be the graph-distance between the endpoints of a SAW π. It is believed (when $d \neq 4$) that there exists an exponent ν and constants $A''_v > 0$, such that

$$\langle\|\pi\|^2\rangle_n^v \sim A''_v n^{2\nu}, \qquad v \in V.$$

As above, this should hold for $d = 4$ subject to the inclusion of the correction factor $(\log n)^{1/4}$. It is believed that $\nu = \frac{1}{2}$ when $d \geq 4$.

The three exponents γ, η, ν are believed to be related through the so-called *Fisher relation*

$$\gamma = \nu(2-\eta).$$

In [17, Sect. 3], reasonable definitions of the three exponents are presented, none of which depend on the existence of embeddings into $\mathbb{R}^d$. Furthermore, it is proved that the values of the exponents are unchanged under the Fisher transformation.

3. Bounds for connective constants

Let G be an infinite, connected, Δ-regular graph. How large or small can $\mu(G)$ be? It is trivial that $\sigma_n(v) \le \Delta(\Delta-1)^{n-1}$, whence $\mu(G) \le \Delta - 1$. It is not difficult to prove the strict inequality $\mu(G) < \Delta - 1$ when G is quasi-transitive and contains a cycle (see [16, Thm 4.2]). Lower bounds are harder to obtain.

Theorem 3.1. [16, Thm 4.1] *Let $\Delta \ge 2$, and let G be an infinite, connected, Δ-regular, vertex-transitive graph. Then $\mu(G) \ge \sqrt{\Delta-1}$ if either*

(a) *G is simple, or*
(b) *G is non-simple and $\Delta \le 4$.*

Note that, for the bridge graph $\mathbb{B}_\Delta$ with $\Delta \ge 2$, we have the equality $\mu(\mathbb{B}_\Delta) = \sqrt{\Delta-1}$. Theorem 3.1(a) answers a question of Itai Benjamini.

Question 3. *What is the best universal lower bound in case* (a) *above? In particular, could it be the case that $\mu(G) \ge \phi$ for any infinite, connected, cubic, vertex-transitive,* simple *graph G?*

Question 4. *Is it the case that $\mu(G) \ge \sqrt{\Delta-1}$ in the non-simple case* (b) *with $\Delta > 4$?*

Here is an outline of the proof of Theorem 3.1. A SAW is called *forward-extendable* if it is the initial segment of some infinite SAW. Let $\sigma_n^{\mathrm{F}}(v)$ be the number of forward-extendable SAWs starting at v. Theorem 3.1 is proved by showing as follows that

$$\sigma_{2n}^{\mathrm{F}}(v) \ge (\Delta-1)^n. \tag{3.1}$$

Let π be a (finite) SAW from v, with final endpoint w. For a vertex $x \in \pi$ satisfying $x \ne w$, and an edge $e \notin \pi$ incident to x, the pair (x, e) is called *π-extendable* if there exists an infinite SAW starting at v whose initial segment traverses π until x, and then traverses e.

First, it is proved subject to a certain condition Π that, for any $2n$-step forward-extendable SAW π, there are at least $n(\Delta - 2)$ π-extendable pairs. Inequality (3.1) may be deduced from this statement.

The second part of the proof is to show that graphs satisfying either (a) or (b) of the theorem satisfy condition Π. It is fairly simple to show that (b) suffices, and it may well be reasonable to extend the conclusion to values of Δ greater than 4.

The growth rate μ_{F} of the number of forward-extendable SAWs has been studied further by Grimmett, Holroyd, and Peres [15]. They show that $\mu_{\mathrm{F}} = \mu$ for any infinite, connected, quasi-transitive graph, with further results involving the numbers of backward-extendable and doubly-extendable SAWs.

4. Strict inequalities for connective constants

4.1. **Outline of results.** Consider a probabilistic model on a graph G, such as a percolation or random-cluster model (see [13]). There is a parameter (perhaps 'density' p or 'temperature' T) and a 'critical point' (usually written p_{c} or T_{c}). The numerical value of the critical point depends on the choice of graph G. It is often important to understand whether a systematic change in the graph causes a *strict* change in the value of the critical point. A general approach to this issue was presented by Aizenman and Grimmett [1] and developed further in [6, 11] and [12, Chap. 3]. The purpose of this section is to review work of [19] directed at the corresponding question for self-avoiding walks.

Let G be a subgraph of G', and suppose each graph is quasi-transitive. It is trivial that $\mu(G) \le \mu(G')$. Under what conditions does the strict inequality $\mu(G) < \mu(G')$ hold? Two sufficient conditions for the strict inequality are presented in [19], and are reviewed here. This is followed in Section 5 with a summary of the consequences for Cayley graphs.

4.2. **Quotient graphs.** Let $G = (V, E)$ be a vertex-transitive graph. Let Γ be a subgroup of the automorphism group $\mathrm{Aut}(G)$ that acts transitively, and let $\mathcal{A}$ be a normal subgroup of Γ (we shall discuss the non-normal case later). There are several ways of constructing a quotient graph $G/\mathcal{A}$, the strongest of which (for our purposes) is given next. The set of neighbours of a vertex $v \in V$ is denoted by ∂v.

We denote by $\vec{G} = (\overline{V}, \vec{E})$ the *directed* quotient graph $G/\mathcal{A}$ constructed as follows. Let $\approx$ be the equivalence relation on V given by $v_1 \approx v_2$ if and only if there exists $\alpha \in \mathcal{A}$ with $\alpha v_1 = v_2$. The vertex-set $\overline{V}$ comprises the equivalence classes of $(V, \approx)$, that is, the orbits $\overline{v} := \mathcal{A}v$ as v ranges over V. For $v, w \in V$, we place $|\partial v \cap \overline{w}|$ directed edges from $\overline{v}$ to $\overline{w}$ (if $\overline{v} = \overline{w}$, these edges are directed loops).

Example 4.1. *As a simple example of a quotient graph, consider the square lattice $G = \mathbb{Z}^2$ and let $m \geq 1$. Let Γ be the set of translations of $\mathbb{Z}^2$, and let $\mathcal{A}$ be the normal subgroup of Γ generated by the map that sends each (i, j) to $(i + m, j)$. The quotient graph $G/\mathcal{A}$ is the square lattice 'wrapped around a cylinder', with each edge replaced by two oppositely directed edges.*

A second example is presented using the language of Cayley graphs in Example 5.2.

Since $\vec{G}$ is obtained from G by a process of identification of vertices and edges, it is natural to ask whether $\mu(\vec{G}) < \mu(G)$. Sufficient conditions for this strict inequality are presented next.

Let $L = L(G, \mathcal{A})$ be the length of the shortest SAW of G with (distinct) endpoints in the same orbit. Thus, for example, $L = 1$ if $\vec{G}$ possesses a directed loop. A group is called *trivial* if it comprises the identity only.

Theorem 4.2. [19, Thm 3.8] *Let Γ act transitively on G, and let $\mathcal{A}$ be a non-trivial, normal subgroup of Γ. The connective constant $\vec{\mu} = \mu(\vec{G})$ satisfies $\vec{\mu} < \mu(G)$ if: either*

(a) *$L \neq 2$, or*
(b) *$L = 2$ and either of the following holds:*
  (i) *G contains some 2-step SAW $v\ (= w_0), w_1, w_2\ (= v')$ satisfying $\overline{v} = \overline{v'}$ and $|\partial v \cap \overline{w_1}| \geq 2$,*
  (ii) *G contains some SAW $v\,(= w_0), w_1, w_2, \dots, w_l\,(= v')$ satisfying $\overline{v} = \overline{v'}$, $\overline{w_i} \neq \overline{w_j}$ for $0 \leq i < j < l$, and furthermore $v' = \alpha v$ for some $\alpha \in \mathcal{A}$ which fixes no w_i.*

Question 5. *In the situation of Theorem 4.2, can one* calculate *an explicit $\epsilon = \epsilon(G, \mathcal{A}) > 0$ such that $\mu(G) - \mu(\vec{G}) > \epsilon$? A partial answer is provided at* [19, Thm 3.11].

We call $\mathcal{A}$ *symmetric* if

$$|\partial v \cap \overline{w}| = |\partial w \cap \overline{v}|, \qquad v, w \in V.$$

Consider the special case $L = 2$ of Theorem 4.2. Condition (i) of Theorem 4.2(b) holds if $\mathcal{A}$ is symmetric, since $|\partial w \cap \overline{v}| \geq 2$. Symmetry of $\mathcal{A}$ is implied by unimodularity, but we prefer to avoid the topic of unimodularity in this review, referring the reader instead to [19, Sect. 3.5] or [31, Sect. 8.2].

Example 4.3. *Conditions (i)–(ii) of Theorem 4.2(b) are necessary in the case $L = 2$, in the sense illustrated by the following example. Let G be the infinite cubic tree with a distinguished end ω. Let Γ be the set* *of automorphisms that preserve ω, and let $\mathcal{A}$ be the normal subgroup generated by the interchange of two children of a given vertex v (and the associated relabelling of their descendants). The graph $\vec{G}$ is isomorphic to that obtained from $\mathbb{Z}$ by replacing each edge by two directed edges in one direction and one in the reverse direction. It is easily seen that $L = 2$, but that neither (i) nor (ii) holds. Indeed, $\mu(\vec{G}) = \mu(G) = 2$.*

The conclusion of Theorem 4.2 is generally invalid under the weaker assumption that $\mathcal{A}$ acts quasi-transitively on G. Consider, for example, the graph G of Figure 4.1, with $\mathcal{A} = \{\iota, \rho\}$ where ρ is reflection in the horizontal axis. Both G and its quotient graph have connective constant 1.

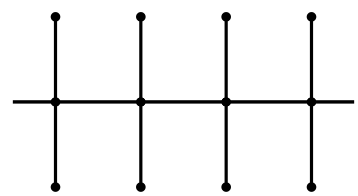

FIGURE 4.1. The pattern is extended infinitely in both directions.

The proof of Theorem 4.2 follows partly the general approach of Kesten in his pattern theorem, see [27] and [32, Sect. 7.2]. Any n-step SAW $\vec{\pi}$ in the directed graph $\vec{G}$ lifts to a SAW π in the larger graph G. The idea is to show that 'most' such $\vec{\pi}$ contain at least an sub-SAWs for which the corresponding sub-walks of π may be replaced by SAWs on G. Different subsets of these sub-SAWs of $\vec{G}$ give rise to different SAWs on G. The number of such subsets grows exponentially in n, and this introduces an exponential 'entropic' factor in the counts of SAWs.

Unlike Kesten's proof and its subsequent elaborations by others, our results apply in the general setting of vertex-transitive graphs, and they utilize algebraic and combinatorial techniques.

We discuss next the assumption of normality of $\mathcal{A}$ in Theorem 4.2. The (undirected) simple quotient graph $\overline{G} = (\overline{V}, \overline{E})$ may be defined as follows even if $\mathcal{A}$ is not a normal subgroup of Γ. As before, the vertex-set $\overline{V}$ is the set of orbits of V under $\mathcal{A}$. Two distinct orbits $\mathcal{A}v$, $\mathcal{A}w$ are declared adjacent in $\overline{G}$ if there exist $v' \in \mathcal{A}v$ and $w' \in \mathcal{A}w$ with $\langle v', w' \rangle \in E$. We write $\overline{G} = G_{\mathcal{A}}$ to emphasize the role of $\mathcal{A}$.

The relationship between the *site percolation* critical points of G and $G_{\mathcal{A}}$ is the topic of a conjecture of Benjamini and Schramm [5], which appears to make the additional assumption that $\mathcal{A}$ acts freely on V. The last assumption is stronger than the assumption of unimodularity.

We ask for an example in which the non-normal case is essentially different from the normal case.

Question 6. *Let* Γ *be a subgroup of* $\mathrm{Aut}(G)$ *acting transitively on* G*. Can there exist a non-normal subgroup* $\mathcal{A}$ *of* Γ *such that:* (i) *the quotient graph* $G_{\mathcal{A}}$ *is vertex-transitive, and* (ii) *there exists no normal subgroup* $\mathcal{N}$ *of some transitively acting* Γ' *such that* $G_{\mathcal{A}}$ *is isomorphic to* $G_{\mathcal{N}}$*? Might it be relevant to assume that* $\mathcal{A}$ *acts freely on* V*?*

We return to connective constants with the following question.

Question 7. *Is it the case that* $\mu(G_{\mathcal{A}}) < \mu(G)$ *under the assumption that* $\mathcal{A}$ *is a non-trivial (not necessarily normal) subgroup of* Γ *acting freely on* V*, such that* $G_{\mathcal{A}}$ *is vertex-transitive?*

The proof of Theorem 4.2 may be adapted to give an affirmative answer to Question 7 subject a certain extra condition on $\mathcal{A}$, see [19, Thm 3.12]. Namely, it suffices that there exists $l \in \mathbb{N}$ such that $G_{\mathcal{A}}$ possesses a cycle of length l but G has no cycle of this length.

4.3. **Quasi-transitive augmentations.** We consider next the systematic addition of new edges, and the effect thereof on the connective constant. Let $G = (V, E)$ be an infinite, connected, vertex-transitive, simple graph. From G, we derive a second graph $G' = (V, E')$ by adding further edges to E, possibly in parallel to existing edges. *We assume that* E *is a proper subset of* E', and introduce next a certain technical property. Let Γ be a subgroup of $\mathrm{Aut}(G)$ that acts transitively.

Definition 4.4. *A subgroup* $\mathcal{A}$ *of* Γ *is said to* have the finite coset property *(relative to* Γ*) with* root $\rho \in V$ *if there exist* $\nu_0, \nu_1, \dots, \nu_s \in \Gamma$*, with* $\nu_0 = \iota$ *and* $s < \infty$*, such that* V *is partitioned as* $\bigcup_{i=0}^{s} \nu_i \mathcal{A}\rho$*. It is said simply to have the finite coset property if it has this property with some root.*

This definition is somewhat technical. The principal situation in which $\mathcal{A}$ has the finite coset property arises, as stated in Theorem 4.7, when $\mathcal{A}$ is a normal subgroup of Γ that acts quasi-transitively on G.

Theorem 4.5. [19, Thm 3.2] *Let* Γ *act transitively on* G*, and let* $\mathcal{A}$ *be a subgroup of* Γ *with the finite coset property. If* $\mathcal{A} \subseteq \mathrm{Aut}(G')$*, then* $\mu(G) < \mu(G')$*.*

Example 4.6. *Let* $\mathbb{Z}^2$ *be the square lattice, with* $\mathcal{A}$ *the group of its translations. The triangular lattice* $\mathbb{T}$ *is obtained from* $\mathbb{Z}^2$ *by adding the edge* $e = \langle 0, (1,1) \rangle$ *together with its images under* $\mathcal{A}$*, where* 0 *denotes the origin. Since* $\mathcal{A}$ *is a normal subgroup of itself with the finite coset* *property, it follows that* $\mu(\mathbb{Z}^2) < \mu(\mathbb{T})$*. This example may be extended to augmentions by other periodic families of new edges, as explained in* [19, Example 3.4].

Question 8. *In the situation of Theorem 4.5, can one calculate* $\epsilon > 0$ *such that* $\mu(G') - \mu(G) > \epsilon$*? (See the related Question 5.)*

Two classes of subgroup $\mathcal{A}$ with the finite coset property are given as follows.

Theorem 4.7. [19, Prop. 3.3] *Let* Γ *act transitively on* G*, and* $\rho \in V$*. The subgroup* $\mathcal{A}$ *of* Γ *has the finite coset property with root* ρ *if either of the following holds.*

1. $\mathcal{A}$ *is a normal subgroup of* Γ *which acts quasi-transitively on* G*.*
2. *The index* $[\Gamma : A]$ *is finite.*

It would be insufficient to assume only quasi-transitivity in Theorem 4.5. Consider, for example, the pair G, G' of Figure 4.2, each of which has connective constant 1.

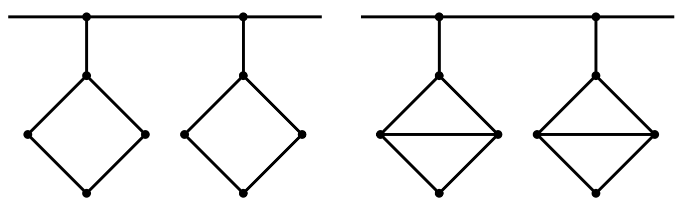

FIGURE 4.2. The pair G, G'. The graphs are extended in both directions. Each graph is quasi-transitive with connective constant 1, and the second is obtained from the first by a quasi-transitive addition of edges.

By Theorem 4.7(a), $\mu(G) < \mu(G')$ if $\mathcal{A}$ is a normal subgroup of some transitive Γ, and $\mathcal{A}$ acts quasi-transitively. Can we dispense with the assumption of normality?

Question 9. *Let* Γ *act transitively on* G*, and let* $\mathcal{A}$ *be a subgroup of* Γ *that acts quasi-transitively on* G*. If* $\mathcal{A} \subseteq \mathrm{Aut}(G')$*, is it necessarily the case that* $\mu(G) < \mu(G')$*?*

A positive answer would be implied by an affirmative answer to the following question. For $\mathcal{A} \subseteq \Gamma \subseteq \mathrm{Aut}(G)$, we say that $\Gamma \setminus \mathcal{A}$ *acts freely on* V if: whenever $\gamma \in \Gamma$ and $v \in V$ satisfy $\gamma v = v$, then $\gamma \in \mathcal{A}$.

Question 10. *Let* G *be a vertex-transitive graph, and let* $\mathcal{A}$ *be a subgroup of* $\mathrm{Aut}(G)$ *that acts quasi-transitively on* G*. Does there exist (or, weaker,* when *does there exist) a subgroup* Γ *of* $\mathrm{Aut}(G)$ *acting transitively on* G *such that* $\mathcal{A} \subseteq \Gamma$ *and* $\Gamma \setminus \mathcal{A}$ *acts freely on* V*?*

See [19, Prop. 3.6] and the further discussion therein.

5. Connective constants of Cayley graphs

Theorems 4.2–4.5 have the following implications for Cayley graphs. Let Γ be an infinite group with a finite generator-set S, where S satisfies $S = S^{-1}$ and $\iota \notin S$. Thus, Γ has a presentation as $\Gamma = \langle S \mid R \rangle$ where R is a set of relators. The Cayley graph $G = G(\Gamma, S)$ is defined as follows. The vertex-set V of G is the set of elements of Γ. Distinct elements $g, h \in V$ are connected by an edge if and only if there exists $s \in S$ such that $h = gs$. It is easily seen that G is connected and vertex-transitive, and it is standard that G is unimodular and hence symmetric. Accounts of Cayley graphs may be found in [3] and [31, Sect. 3.4].

Let $s_1 s_2 \cdots s_l = \iota$ be a relation. This relation corresponds to the closed walk $(\iota, s_1, s_1 s_2, \dots, s_1 s_2 \cdots s_l = \iota)$ of G passing through the identity ι. Consider now the effect of adding a further relator. Let $t_1, t_2, \dots, t_l \in S$ be such that $\rho := t_1 t_2 \cdots t_l$ satisfies $\rho \neq \iota$, and write $\Gamma_\rho = \langle S \mid R \cup \{\rho\} \rangle$. Then Γ_ρ is isomorphic to the quotient group $\Gamma/\mathcal{N}$ where $\mathcal{N}$ is the normal subgroup of Γ generated by ρ.

Theorem 5.1. [19, Corollaries 4.1, 4.3] *Let $G = G(\Gamma, S)$ be the Cayley graph of the infinite, finitely generated group $\Gamma = \langle S \mid R \rangle$.*

(a) *Let $G_\rho = G(\Gamma_\rho, S)$ be the Cayley graph obtained by adding to R a further non-trivial relator ρ. Then $\mu(G_\rho) < \mu(G)$.*
(b) *Let $w \in \Gamma$ satisfy $w \neq \iota$, $w \notin S$, and let $\overline{G}_w$ be the Cayley graph of the group obtained by adding w (and w^{-1}) to S. Then $\mu(G) < \mu(\overline{G}_w)$.*

Example 5.2. *The* square/octagon *lattice, otherwise known as the Archimedean lattice $(4, 8^2)$, is the Cayley graph of the group with generator set $S = \{s_1, s_2, s_3\}$ and relators*

$$\{s_1^2, s_2^2, s_3^2, s_1 s_2 s_1 s_2, s_1 s_3 s_2 s_3 s_1 s_3 s_2 s_3\}.$$

(See [19, Fig. 3].) Adding the further relator $s_2 s_3 s_2 s_3$, we obtain a graph isomorphic to the ladder graph of Figure 1.2, whose connective constant is the golden mean $\phi := \frac{1}{2}(\sqrt{5} + 1)$.

By Corollary 5.1(a), the connective constant μ of the square/octagon lattice is strictly greater than $\phi = 1.618\dots$. The best lower bound currently known appears to be $\mu > 1.804\dots$, see [25].

Example 5.3. *The square lattice $\mathbb{Z}^2$ is the Cayley graph of the abelian group with $S = \{a, b\}$ and $R = \{aba^{-1}b^{-1}\}$. We add a generator ab*

(and its inverse), thus adding a diagonal to each square of $\mathbb{Z}^2$. Theorem 5.1(b) implies the standard inequality $\mu(\mathbb{Z}^2) < \mu(\mathbb{T})$ of Example 4.6.

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