RANDOMLY ORIENTED LATTICES

Geoffrey R. Grimmett

Abstract. The square lattice is used to generate an oriented graph in which a rightward or upward arrow is present on each edge with probability a, and a leftward or downward arrow with probability b. Independence between different edges of the square lattice is assumed, but nothing is assumed concerning the dependence between the two possible orientations at any given edge. A property of self-duality is exploited to show that, when $a + b = 1$, the process is, in a sense to be made precise, either critical or supercritical but not subcritical. This observation enables progress with the percolation problem in which each horizontal edge is oriented rightwards with probability p and otherwise leftwards, and each vertical edge is oriented upwards with probability p and otherwise downwards.

1. Randomly oriented square lattice

Despite the relative maturity of percolation theory (see [4]) there remain a number of specific problems of charm and apparent difficulty. One such is the following. Consider the square lattice $\mathbb{Z}^2$ and let $0 \le p \le 1$. Each horizontal edge is oriented rightwards with probability p, and otherwise leftwards. Each vertical edge is oriented upwards with probability p, and otherwise downwards. Let $\theta(p)$ denote the probability that the origin 0 is the endpoint of an infinite self-avoiding path which is oriented away from 0. The challenge is to determine for which p it is the case that $\theta(p) > 0$.

It is elementary that $\theta(0) = 1$ and that $\theta(p) = \theta(1-p)$. It is less obvious that $\theta(\frac{1}{2}) = 0$; this was observed in Section 10.10 of the first edition of [4]. By a comparison with oriented percolation (see [4, Section 12.8]), we have that $\theta(p) > 0$ if $p > \vec{p}_c$, where $\vec{p}_c$ is the critical point of oriented percolation on $\mathbb{Z}^2$; it is not difficult to prove by the method of [1] that there exists $p' \in (\frac{1}{2}, \vec{p}_c)$ such that $\theta(p) > 0$ when $p > p'$. It is believed that $\vec{p}_c \sim 0.64$, and proved that $\vec{p}_c < 0.6735$; see [6, p. 186] and [2] respectively.

It has been conjectured (in the first edition of [4], and possibly elsewhere) that $\theta(p) > 0$ if $p \neq \frac{1}{2}$, and this note is intended as an explanation, but not a full resolution, of this conjecture. In summary, we will show that, for all p, the above process is either critical or supercritical in the following sense: if any small

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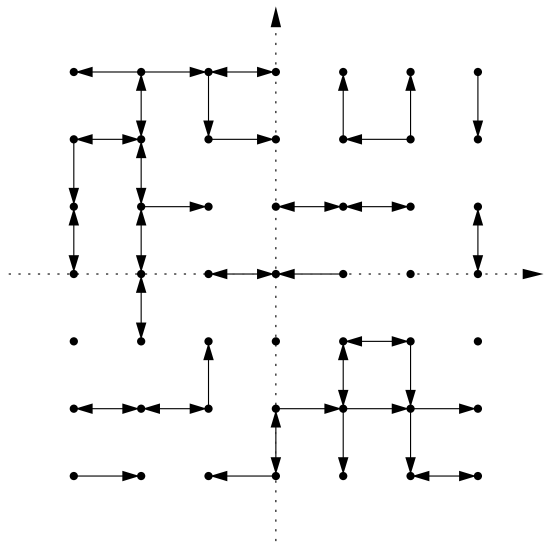

Figure 1. Arcs are placed randomly between pairs of neighbours of the square lattice.

positive density of oriented edges is added at random, then there is a strictly positive probability that the origin is the endpoint of an infinite self-avoiding oriented path.

Although the problem studied in this paper is rather specific, it has several features common to a circle of problems involving graphs with oriented rather than unoriented edges. The now standard technology of unoriented percolation is not by itself enough to answer such questions, and new ideas are needed.

The basic process of this paper is as follows. Let $\rho, \lambda, \beta \geq 0$ and $\rho+\lambda+\beta \leq 1$. For each horizontal edge e of the square lattice, we designate e to be:

oriented only *rightwards*, with probability ρ,
oriented only *leftwards*, with probability λ,
oriented both *rightwards and leftwards*, with probability β,
absent, with probability $1-\rho-\lambda-\beta$.

Each vertical edge is treated similarly, replacing rightwards by upwards, and replacing leftwards by downwards. Different edges receive independent designations. A sample realization of the resulting (random) directed graph is presented in Figure 1. We call the process constructed above the *basic process* with parameters ρ, λ, β.

Let $\theta(\rho, \lambda, \beta)$ denote the probability that the origin 0 is the endpoint of some infinite self-avoiding path oriented away from 0. It has been noted earlier that $\theta(\rho, \lambda, \beta)$ equals the probability that the origin is the endpoint of some infinite self-avoiding path oriented *towards* the origin. Perhaps the easiest way to see this is to reflect the lattice in the line $x_1 + x_2 = 0$ and to reverse all orientations.

A case of special interest arises when rightward and leftward (respectively, upward and downward) arcs are placed independently between each pair of horizontal (respectively, vertical) neighbours. That is, if e is the horizontal edge with endpoints x and y, we place a rightward arc between x and y with probability a, and *independently* a leftward arc with probability b. Vertical edges are treated similarly, replacing rightward/leftward by upward/downward. In this case, we have

$$\rho = a(1-b), \quad \lambda = b(1-a), \quad \beta = ab,$$

so that there exists an infinite oriented self-avoiding path from 0 with probability $\theta(a(1-b), b(1-a), ab)$. We call this process the *independent process* with parameters a, b.

Our first theorem implies that $\theta(\rho, \lambda, \beta)$ depends only on the marginal probabilities $a = \rho + \beta$, $b = \lambda + \beta$ of rightward or leftward (respectively, upward or downward) edges.

Theorem 1.1. *We have that* $\theta(\rho, \lambda, \beta) = \theta(a(1-b), b(1-a), ab)$ *where* $a = \rho + \beta$ *and* $b = \lambda + \beta$.

Theorem 1.2. *If* $a+b > 1$*, the independent process with parameters* a*,* b *contains an infinite oriented self-avoiding path from* 0 *with strictly positive probability, which is to say that* $\theta(a(1-b), b(1-a), ab) > 0$.

Returning to the process described at the beginning of this section, we set $\rho = p$, $\lambda = 1-p$, $\beta = 0$, to obtain $a = \rho + \beta = p$, $b = \lambda + \beta = 1-p$. Note that $a + b = 1$. We have from Theorem 1.1 that $\theta(p, 1-p, 0) = \theta(a(1-b), b(1-a), ab)$. By Theorem 1.2, if we augment the process by adding, independently at random, a positive density ϵ of edges oriented rightwards/upwards, then the resulting random directed graph contains, with a strictly positive probability, an infinite oriented self-avoiding path from 0.

Theorem 1.2, and more, will be proved in Section 4. In Section 2 we prove Theorem 1.1. We explore duality in Section 3, and exponential decay in Section 4.

Here is some notation. The square lattice $\mathbb{Z}^2$ has vertices $\{x = (x_1, x_2) : x_1, x_2 \in \mathbb{Z}\}$ where $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$. We write

$$|x| = |x_1| + |x_2| \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{Z}^2,$$

and we place an edge, denoted $\langle x, y\rangle$, between any pair x, y satisfying $|x - y| = 1$. We shall make use of the box $\Lambda_n = \{x \in \mathbb{Z}^2 : |x| \le n\}$ and its *boundary* $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$.

We shall generally distinguish between an *edge* of $\mathbb{Z}^2$ and an *arc*, the latter term being reserved for an oriented edge joining two neighbours of $\mathbb{Z}^2$. Arcs may be oriented in one or both of the possible directions. Once arcs have been placed on $\mathbb{Z}^2$, we say that vertex y is *reachable* from vertex x, written $x \to y$, if there exists a path of arcs from x to y which is oriented away from x. For sets X, Y of vertices, we write $X \to Y$ if there exists $x \in X$ and $y \in Y$ such that $x \to y$. We write $X \nrightarrow Y$ for the negation of $X \to Y$, and '$X \to Y$ in Z' if there exists an oriented path from some x ($\in X$) to some y ($\in Y$) using only vertices of Z.

2. Marginal probabilities

Consider the basic process with parameters ρ, λ, β, and write $\mathbb{P}$ for the associated probability measure. If $x = (m, n)$ and $y = (m+1, n)$ the chance that there exists an rightward (respectively, leftward) edge joining the pair x, y is $a = \rho + \beta$ (respectively, $b = \lambda + \beta$); a similar statement is valid with $x = (m, n)$, $y = (m, n+1)$, and with rightward/leftward replaced by upward/downward. We refer to a and b as the *marginal* probabilities associated with the basic process with parameters ρ, λ, β.

Proposition 2.1. *For all sets C of vertices, and all $A, B \subseteq C$, the probability $\mathbb{P}(A \to B \text{ in } C)$ depends only on the marginal probabilities $a = \rho+\beta$ and $b = \lambda+\beta$. That is, there exists a function $f = f_{A,B,C}$ such that*

$$\mathbb{P}(A \to B \text{ in } C) = f(\rho+\beta, \lambda+\beta) \quad \textit{for all } \rho, \lambda, \beta.$$

Proof. The proof follows a standard construction (see, for example, [4, pp. 28, 211]), and is therefore given in an abbreviated form only. We order the edges of the lattice in some deterministic manner. We then build up the set of vertices z reachable from A along oriented paths, but examining appropriate edges chosen sequentially according to this predetermined lexicographic ordering. Each time an edge is examined, there exists a unique orientation of that edge which will enable an extension of the set already constructed. Such an edge may be used to extend the set if and only if it is oriented in the prescribed direction. The probability that this occurs is the marginal probability associated with that direction. The probability that this construction ultimately includes some vertex of B depends therefore only on the marginal probabilities. □

Proof of Theorem 1.1. Take C to be the set of all vertices of $\mathbb{Z}^2$, $A = \{0\}$, and $B = \partial\Lambda_n$. By Proposition 2.1, $\mathbb{P}(0 \to \partial\Lambda_n)$ depends only on the marginals $a = \rho + \beta$, $b = \lambda + \beta$. The statement of the theorem follows on passing to the limit as $n \to \infty$. □

Proposition 2.1 and Theorem 1.1 may be generalized to lattices other than the square lattice, and to certain more general probability distributions.

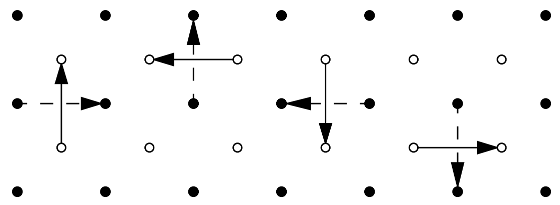

Figure 2. The solid vertices are those of $\mathbb{Z}^2$, and the open vertices are those of the dual $\mathbb{Z}^2_{\rm d}$. If an edge e of the square lattice fails to have an orientation in direction θ, then its dual edge $e_{\rm d}$ has an orientation in direction $\theta + \frac{1}{2}\pi$. Dashed arrows indicate missing orientations in the primal lattice, and solid arrows indicate orientations present in the dual.

3. Duality

We explore the property of duality in this section. A related duality was observed in [3]. Since we are interested mainly in the existence or not of oriented paths with given endpoints, we may in the light of Proposition 2.1 restrict ourselves to the independent model, in which arcs of differing orientations are placed at each e *independently* of one another. Note however that the basic model also has a dual model.

Consider then the independent model in which each pair of 'horizontal' (respectively, 'vertical') neighbours is joined by a rightward (respectively, upward) arc with probability a, and by a leftward (respectively, downward) arc with probability b; all such oriented arcs are placed independently of one another. This is the basic model with $\rho = a(1-b)$, $\lambda = b(1-a)$, $\beta = ab$.

The dual lattice is denoted $\mathbb{Z}^2_{\rm d}$, and is a copy of $\mathbb{Z}^2$ translated by the vector $(\frac{1}{2}, \frac{1}{2})$. The dual origin is $0_{\rm d} = (\frac{1}{2}, \frac{1}{2})$. Each edge of e is traversed by a unique edge $e_{\rm d}$ whose orientation is given as follows: if e fails to have an orientation in direction θ, then $e_{\rm d}$ has an orientation in direction $\theta + \frac{1}{2}\pi$. This rule is illustrated in Figure 2, and, subject to a rotation of the dual lattice $\mathbb{Z}^2_{\rm d}$, leads to an independent model on $\mathbb{Z}^2_{\rm d}$ with parameters $1-a$, $1-b$.

It follows that, when $a + b = 1$, the model is self-dual, but this observation calls for a rigorous exploitation. Let $D(n)$ be the subgraph of $\mathbb{Z}^2$ induced by the set of vertices within the 'offset diamond' $\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2 - \frac{1}{2}| \le n + \frac{1}{2}\}$. The graph $D(n)$ is illustrated in Figure 3, together with a certain subgraph $D(n)_{\rm d}$ of the dual lattice. Note that $D(n)$ and $D(n)_{\rm d}$ are isomorphic graphs.

Let U_n (respectively, V_n) denote the upper left side (respectively, lower right side) of $D(n)$; that is, U_n is the set of vertices $x = (x_1, x_2)$ of $D(n)$ such that $-x_1 + (x_2 - \frac{1}{2}) = n + \frac{1}{2}$, and V_n is the set of x such that $x_1 - (x_2 - \frac{1}{2}) = n + \frac{1}{2}$. In an analogous way, we define the top right side R_n and bottom left side of L_n of $D(n)_{\rm d}$.

Lemma 3.1. *In the independent model with parameters a and b,*

$$\mathbb{P}\big(U_n \to V_n \text{ in } D(n)\big) + \mathbb{P}\big(L_n \to R_n \text{ in } D(n)_{\rm d}\big) = 1$$

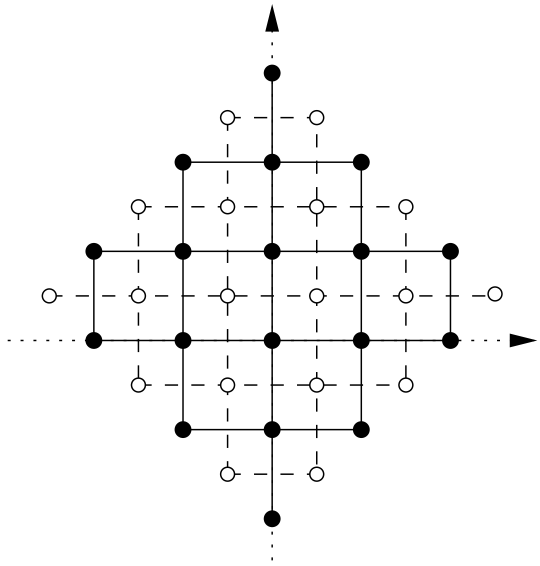

Figure 3. The solid lines are the edges of the subgraph $D(n)$ of $\mathbb{Z}^2$ when $n = 2$. The dashed lines are those of the associated 'dual' graph $D(n)_{\mathrm{d}}$ of $\mathbb{Z}^2_{\mathrm{d}}$.

for all $n \geq 0$.

Proof. This is achieved in a way similar to that used for bond percolation on $\mathbb{Z}^2$; see [9, 10] and [4, p. 294]. First, in rough terms, only one of the events $\{U_n \to V_n \text{ in } D(n)\}$, $\{L_n \to R_n \text{ in } D(n)_{\mathrm{d}}\}$ may occur, since if both occur, there necessarily exists an edge e with dual e_{d} the orientations of which contradict the rule given above.

Secondly, if $U_n \not\to V_n$ in $D(n)$ then the 'edge boundary' of the set of vertices reachable from U_n contains, in its dual, an oriented path from L_n to R_n. □

We write

$$g_n(a, b) = \mathbb{P}\big(U_n \to V_n \text{ in } D(n)\big)$$

and we deduce from Lemma 3.1 with attention to the order of the arguments of g_n that

$$g_n(a, b) + g_n(1 - a, 1 - b) = 1 \quad \text{for all } n \geq 0. \tag{3.1}$$

Lemma 3.2. *We have for all* a, b, n *that*

$$g_{n+1}(b, a) \geq ab g_n(a, b), \tag{3.2}$$

$$g_{n+1}(a, b) \geq \max\{a^2, b^2\} g_n(a, b). \tag{3.3}$$

Proof. We may augment $D(n)$ to a larger graph $D(n)'$ induced by the set of vertices within the diamond $\{(x_1, x_2) : |x_1 - \frac{1}{2}| + |x_2| \leq n + \frac{3}{2}\}$; $D(n)'$ is illustrated in Figure 4. If $U_n \to V_n$ in $D(n)$, we may pick an oriented path π of $D(n)$ joining some

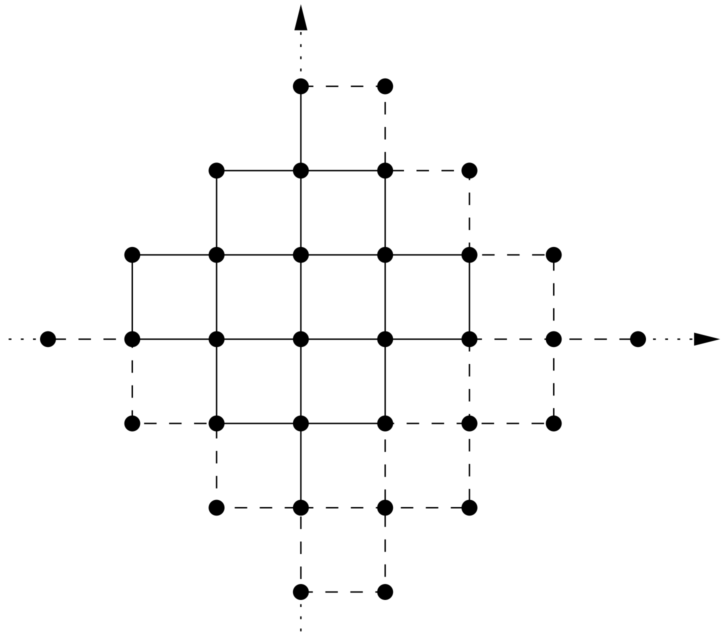

Figure 4. The region $D(n)$ together with the enlarged region $D(n)'$ obtained by adding the dashed edges around part of the boundary of $D(n)$.

$u \in U_n$ to some $v \in V_n$. Let E_v be the event that $\langle v, v+e_1\rangle$ and $\langle v+e_1, v+e_1-e_2\rangle$ are oriented in the respective directions v to $v+e_1$, and $v+e_1$ to $v+e_1-e_2$. (Here, $e_1=(1,0)$ and $e_2=(0,1)$.) Conditional on the choice of v, the event E_v is independent of $\{U_n \to V_n \text{ in } D(n)\}$, and the conditional probability of E_v is ab. It follows that

$$\mathbb{P}\big(U_n \to V_n' \text{ in } D(n)'\big) \geq ab\mathbb{P}\big(U_n \to V_n \text{ in } D(n)\big)$$

where V_n' is the bottom right side of $D(n)'$. Inequality (3.2) follows. Inequality (3.3) may be proved in a similar way, using the enlargement $D(n)''$ of $D(n)$ defined as the graph induced by the set of vertices within the diamond $\{(x_1,x_2) : |x_1| + |x_2 - \frac{1}{2}| \leq n + \frac{3}{2}\}$; $D(n)''$ is illustrated in Figure 5. □

Combining (3.1) and (3.2), we obtain the following inequality:

$$(1-a)(1-b)g_n(a,b) + g_{n+1}(1-b,1-a) \geq (1-a)(1-b) \tag{3.4}$$

for all $0 \leq a,b \leq 1$ and $n \geq 0$. In particular,

$$a(1-a)g_n(a,1-a) + g_{n+1}(a,1-a) \geq a(1-a) \tag{3.5}$$

for all $0 \leq a \leq 1$ and $n \geq 0$. Using (3.3) we obtain that

$$\left(\frac{a(1-a)}{\max\{a^2,(1-a)^2\}} + 1\right) g_{n+1}(a,1-a) \geq a(1-a),$$

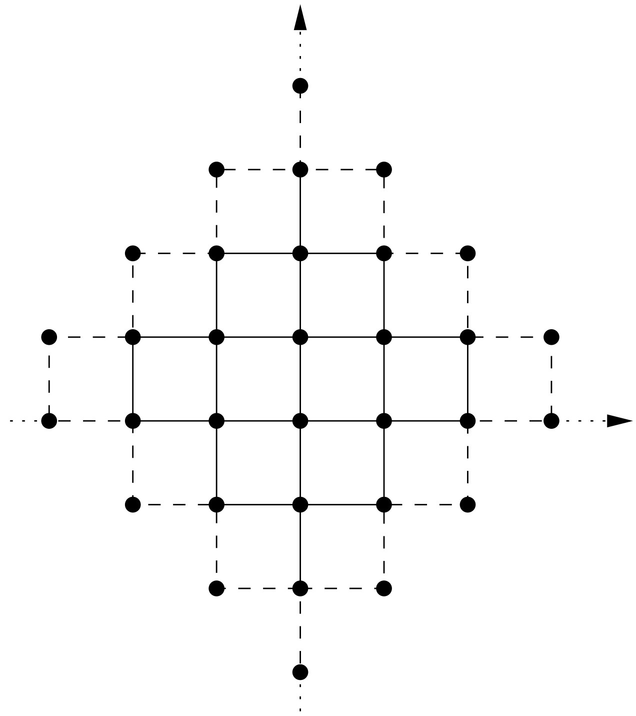

Figure 5. The region $D(n)$ together with the enlarged region $D(n)''$ obtained by adding the dashed edges around part of the boundary of $D(n)$.

and we arrive at the following.

Lemma 3.3. *We have that*

$$g_n(a, 1-a) \geq a(1-a)\max\{a, 1-a\} \quad \textit{for all } a \textit{ and } n \geq 1.$$

It is reasonable to guess that the limit

$$g(a, 1-a) = \lim_{n\to\infty} g_n(a, 1-a)$$

exists, but we do not prove this here.

4. Exponential decay

Consider the independent model with marginal probabilities a and b, and let $\psi(a,b) = \theta(a(1-b), b(1-a), ab)$ be the probability that there exists an infinite self-avoiding path oriented away from its endpoint 0. We write $\mathbb{P}_{a,b}$ for the associated probability measure, and write $\Lambda_n = \{x \in \mathbb{Z}^2 : |x| \leq n\}$ and $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$ as before.

It is easily seen that $\psi(a,b) = 0$ if and only if, $\mathbb{P}_{a,b}$-a.s., there exists an anticlockwise circuit in the dual having 0 in its inside. Using the usual circuit-counting argument (see [4, p. 17]), we obtain that

$$\psi(a,b) > 0 \quad \text{if} \quad (1-a)(1-b) < \frac{1}{\mu^2}$$

where μ is the connective constant of $\mathbb{Z}^2$. We recall the estimate $\mu \sim 2.64$; see [5, p. 482].

Theorem 4.1. *Assume that $\psi(a,b) = 0$ and that $0 \le a' < a$, $0 \le b' < b$. There exists a strictly positive quantity $c = c(a', b')$ such that*

$$\mathbb{P}_{a',b'}(0 \to \partial\Lambda_n) \le e^{-cn} \quad \textit{for all} \;\; n \ge 1.$$

Proof. This may be proved in exactly the same way as Menshikov's proof of exponential decay for subcritical percolation (see [7, 8], [4, Section 5.2]). The relevant properties of the measures $\mathbb{P}_{\alpha,\beta}$ are: invariance under translations of $\mathbb{Z}^2$, stochastic monotonicity in α and β, and the fact that they are product measures on the appropriate product of copies of $\{0, 1\}$. The detailed proof contains nothing new of substance, and is omitted. □

Proof of Theorem 1.2. Let $\rho, \lambda, \beta \ge 0$ and $\rho + \lambda + \beta \le 1$, and set $a = \rho + \beta$, $b = \lambda + \beta$. Assume we may pick ρ, λ, β such that $a + b > 1$ and $\theta(\rho, \lambda, \beta) = 0$. Since $a, b \le 1$, it must be the case that $a, b > 0$. We have from Theorem 1.1 that

$$\theta(\rho, \lambda, \beta) = \mathbb{P}_{a,b}(0 \to \infty) = \psi(a, b).$$

Set

$$a' = \frac{a}{a+b}, \quad b' = \frac{b}{a+b},$$

so that $a' < a$ and $b' < b$, and $a' + b' = 1$. By Lemma 3.3,

$$g_n(a', b') \ge \tfrac{1}{2} a'b' \quad \text{for } n \ge 1. \tag{4.1}$$

By Theorem 4.1 and the fact that $|U_n| = n + 1$, there exists $c > 0$ such that

$$g_n(a', b') \le (n+1)e^{-cn}. \tag{4.2}$$

Inequalities (4.1) and (4.2) provide a contradiction for large n, and we deduce that no such triple ρ, λ, β exists. □

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References

1. Aizenman, M. and Grimmett, G. R., *Strict monotonicity for critical points in percolation and ferromagnetic models*, Journal of Statistical Physics **63** (1991), 817–835.
2. Balister, P., Bollobás, B., and Stacey, A., *Improved upper bounds for the critical probability in oriented percolation*, Random Structures and Algorithms **5** (1994), 573–589.

3. Dhar, D., Barma, M., and Phani, M. K., *Duality transformations for two-dimensional directed percolation and resistance problems*, Physical Review Letters **47** (1981), 1238–1241.
4. Grimmett, G. R., *Percolation*, 2nd edition, Springer-Verlag, Berlin, 1999.
5. Hughes, B. D., *Random Walks and Random Environments, Volume 1: Random Walks*, Oxford University Press, Oxford, 1995.
6. Hughes, B. D., *Random Walks and Random Environments, Volume 2: Random Environments*, Oxford University Press, Oxford, 1996.
7. Menshikov, M. V., *Coincidence of critical points in percolation problems*, Soviet Mathematics Doklady **33** (1986), 856–859.
8. Menshikov, M. V., Molchanov, S. A., and Sidorenko, A. F., *Percolation theory and some applications*, Itogi Nauki i Techniki (Series of Probability Theory, Mathematical Statistics, Theoretical Cybernetics) **24** (1986), 53–110.
9. Russo, L., *A note on percolation*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete **43** (1978), 39–48.
10. Seymour, P. D. and Welsh, D. J. A., *Percolation probabilities on the square lattice*, Advances in Graph Theory (B. Bollobás, ed.), Annals of Discrete Mathematics 3, North-Holland, Amsterdam, 1978, pp. 227–245.

Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, United Kingdom

E-mail address: g.r.grimmett@statslab.cam.ac.uk

URL: http://www.statslab.cam.ac.uk/~grg/