Symbols

Symbol Description

Symbol	Description
$Z_{\mathrm{I}}(G)$	The Ising partition function
$Z_{\mathrm{P}}(G)$	The Potts partition function
$Z_{\mathrm{RC}}(G)$	The random-cluster partition function
UCS	Uniform connected subgraph
UST	Uniform spanning tree
USF	Uniform spanning forest

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The Potts and random-cluster models

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CONTENTS

1.1 Synopsis

The relationship between the Tutte polynomial and the random-cluster, Ising, and Potts models of statistical physics is summarised. Certain fundamental properties of these models are described, particularly those that may be expressed neatly in terms of Tutte polynomials. The flow and chromatic polynomials feature naturally in the study of the Potts model.

- Ising, Potts, and random-cluster models; physical origins; couplings, partition functions.
- Basic properties of random-cluster measures; stochastic ordering; comparison inequalities, positive association.
- Limit as $q \downarrow 0$; uniform spanning tree, uniform spanning forest, uniform connected subgraph; negative association.
- Flow polynomial; Potts two-point correlation; Simon inequality.
- Zero-temperature limit; chromatic polynomial.
- Asymptotics of the Tutte polynomial on the complete graph.

1.2 Introduction

The four principal elements in this chapter are the Ising model of 1925, the Tutte polynomial of 1947, the Potts model of 1952, and the random-cluster model of 1972.

The *Ising model*[1] [28] is the fundamental model for the ferromagnet, and it has generated enormous interest and activity in mathematics and physics over the intervening decades. The *Potts model* [36] extends the number of local states of the Ising model from 2 to a general number q. The *random-cluster model* of Fortuin and Kasteleyn [20] provides an overarching framework for the Ising/Potts models that incorporates percolation and electrical networks, together with certain other processes. The common aspect of importance for these three systems is the singularity that occurs at points of phase transition.

Whereas these three processes originated in mathematical physics, the *Tutte polynomial* [44] is an object from combinatorics, and it encapsulates a number of significant features of a graph or matroid. It turns out that the Tutte polynomial is equal (subject to a change of variables) to the partition

[1] The Ising model was proposed to Ising by Lenz, and the Potts model to Potts by Domb.

function of the random-cluster model, and therefore to that of the Potts model. This connection is not a coincidence since both the Tutte and random-cluster functions arose in independent analyses of local graph operations such as deletion and contraction.

The Tutte polynomial originated in Tutte's exploration of deletion and contraction on a finite graph. The random-cluster model originated similarly in Kasteleyn's observation that the Ising, Potts, and percolation models, and also electrical networks, have a property of invariance under series and parallel operations on edges.

Combinatorial theory and statistical mechanics are areas of science which have much in common, while retaining their distinctive characteristics. Statistical mechanics is mostly concerned with the structure of phases and of singularities, and has developed appropriate methodology and language. Although, in principle, the properties of a physical model are encoded entirely within its partition function, the extraction of such properties is often challenging and hinges frequently on other factors such as the nature of the underlying graph.

The connection between the Tutte polynomial and statistical mechanics is summarised in this chapter, as follows. We aim: (i) to give a clear formulation of the relevant models, (ii) to explain the connection between their partition functions and the Tutte polynomial, (iii) to present some of the basic properties of the random-cluster model that are contingent on the partition function, and (iv) to present a selection of open problems concerning Potts and random-cluster models that may be related to the Tutte polynomial.

Further references containing material relevant to this chapter include [5, 10, 16, 25, 42, 45, 46, 47, 48].

1.3 Probabilistic Models from Physics

1.3.1 The Ising and Potts ferromagnets

The Ising model for ferromagnetism was analysed in one dimension in Ising's thesis and 1925 paper [28]. It modelled the following physical experiment. A piece of iron is placed in a magnetic field, with an intensity that is increased from zero to a maximum, and then reduced to zero. The iron retains some residual magnetisation if and only if the temperature is sufficiently low, and the critical temperature for this phenomenon is called the *Curie point*.

The Ising model may be summarised as follows. Suppose that particles are placed at the vertices of a graph embedded in a Euclidean space. Each particle may be in either of two states: spin 'up' or spin 'down'. Spin-values are chosen at random according to a certain probability measure governed by interactions between neighbouring particles. This measure is described as follows.

Let $G = (V, E)$ be a finite, simple graph. Each vertex $v \in V$ is occupied by a particle with a random spin. Since spins are assumed to come in two basic types, we take as sample space the set $\Sigma = \{-1, +1\}^V$ of vectors $\sigma = (\sigma_x : x \in V)$ with entries ± 1.

Definition 1.3.1. Let $\beta \in (0, \infty)$ and $h \in \mathbb{R}$. The Ising *Hamiltonian* H_{I} and *partition function* Z_{I} are given by

$$H_{\mathrm{I}}(\sigma) = -\sum_{e=\langle x,y\rangle \in E} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x, \qquad Z_{\mathrm{I}}(G; \beta, h) = \sum_{\sigma \in \Sigma} e^{-\beta H_{\mathrm{I}}(\sigma)}. \quad (1.1)$$

The Ising probability measure $\lambda_{\beta,h}$ on G is defined by

$$\lambda_{\beta,h}(\sigma) = \frac{1}{Z_{\mathrm{I}}} e^{-\beta H_{\mathrm{I}}(\sigma)}. \qquad \sigma \in \Sigma,$$

The parameter β represents the reciprocal $1/T$ of temperature, and h is the *external field*. The second summation of (1.1) may be subsumed into the first by adding a new 'ghost' vertex to the graph, and connecting it to each member of V. Such an augmentation is a classical device in the study of the Ising model.

For reasons of simplicity, we shall assume generally here that $h = 0$, and we write $\lambda_\beta = \lambda_{\beta,0}$. It is usual to include also an edge-interaction J, which we have chosen to absorb into the parameter β. The above model is called *ferromagnetic* (in that $\beta > 0$) in contrast to the antiferromagnet of Section 1.3.5.

The Ising model has two admissible spin-values, and a very rich theory. In his 1952 paper [36], Potts developed an extension of the Ising model to a general number of spin-values.

Let q be an integer satisfying $q \geq 2$, and consider the sample space $\Sigma = \{1, 2, \dots, q\}^V$. Each vertex of G may now be in any of q states.

Definition 1.3.2. Let $\beta \in (0, \infty)$ and $q \in \{2, 3, \dots\}$. The Potts Hamiltonian H_{P} and partition function Z_{P} are given by

$$H_{\mathrm{P}}(\sigma) = -\sum_{e=\langle x,y\rangle} \delta_{\sigma_x,\sigma_y}, \qquad Z_{\mathrm{P}}(G; \beta, q) = \sum_{\sigma \in \Sigma} e^{-\beta H_{\mathrm{P}}(\sigma)},$$

where $\delta_{u,v}$ is the Kronecker delta. The Potts probability measure on G is defined by

$$\pi_{\beta,q}(\sigma) = \frac{1}{Z_{\mathrm{P}}} e^{-\beta H_{\mathrm{P}}(\sigma)}, \qquad \sigma \in \Sigma.$$

When $q = 2$, we have that

$$\delta_{\sigma_x,\sigma_y} = \tfrac{1}{2}(1 + \sigma_x \sigma_y),$$

from which it follows that the $q = 2$ Potts model is simply the Ising model with β replaced by $\frac{1}{2}\beta$.

In a more general definition, one may include a non-zero external field h and a vector $\mathbf{J}$ of edge-parameters. See Section 1.3.5.

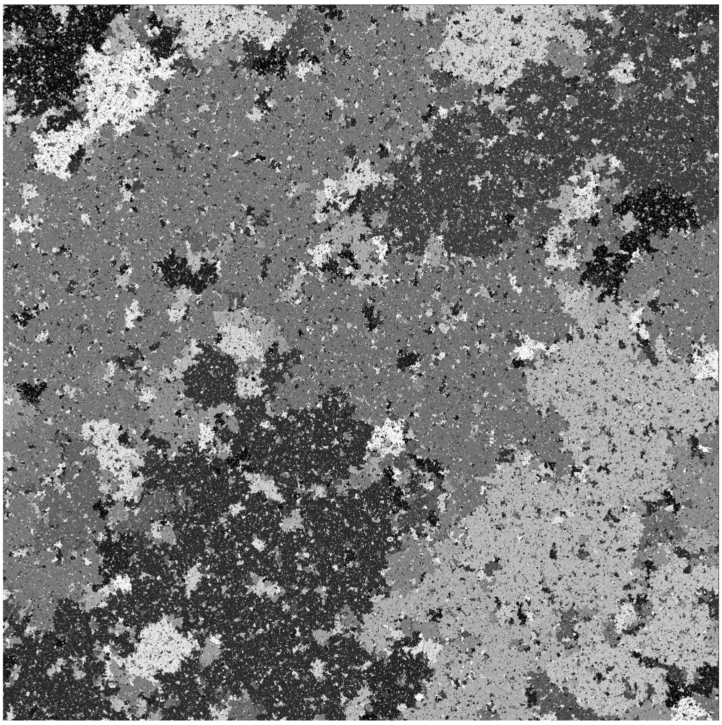

FIGURE 1.1
A sample from the random-cluster model on a 2048×2048 box of the square lattice, with $p = 0.585816$ and $q = 2$. It was obtained by simulating the Ising model and applying the coupling of Section 1.3.3. Each individual cluster is coloured with a different tint of gray, and the smaller clusters are not visible in the picture. (Figure by courtesy of Raphaël Cerf.)

1.3.2 The random-cluster model

The random-cluster model was formulated in a series of papers [18, 19, 20] by Fortuin and Kasteleyn. It is described next, and its relationship to the Potts model is explained in Section 1.3.3.

Let $G = (V, E)$ be a finite, simple graph. The relevant state space is the set $\Omega = \{0, 1\}^E$ of vectors $\omega = (\omega(e) : e \in E)$ with entries 0 or 1. An edge e as said to be *open* in $\omega \in \Omega$ if $\omega(e) = 1$, and *closed* if $\omega(e) = 0$. For $\omega \in \Omega$, let $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ denote the set of open edges, and let $k(\omega)$ be the number of connected components (or 'open clusters') of the graph $(V, \eta(\omega))$; the count $k(\omega)$ includes the number of isolated vertices.

Definition 1.3.3. Let $p \in (0, 1)$ and $q \in (0, \infty)$. The *random-cluster measure*

$\phi_{p,q}$ on G is given by

$$\phi_{p,q}(\omega) = \frac{1}{Z_{\mathrm{RC}}}\left\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega)}, \qquad \omega \in \Omega,$$

where the partition function Z_{RC} is given by

$$Z_{\mathrm{RC}}(G;p,q) = \sum_{\omega\in\Omega}\left\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega)}. \tag{1.2}$$

The most important values of q are arguably the positive integers. When $q = 1$, we have that $\phi_p := \phi_{p,1}$ is a product measure, and the words 'percolation' and 'random graph' are often used in this context, see [22, 29]. The random-cluster model with $q \in \{2, 3, \dots\}$ corresponds, as sketched in the next section, to the Potts model with q local states.

See [23] for the general theory of the random-cluster model.

1.3.3 Coupling of the Potts and random-cluster measures

Let $q \in \{2, 3, \dots\}$, $p \in (0, 1)$, and let $G = (V, E)$ be a finite, simple graph. We consider the product sample space $\Sigma \times \Omega$ where $\Sigma = \{1, 2, \dots, q\}^V$ and $\Omega = \{0, 1\}^E$ as above. Let μ be the probability measure on $\Sigma \times \Omega$ given by

$$\mu(\sigma, \omega) \propto \psi(\sigma)\phi_p(\omega)1_F(\sigma, \omega), \qquad (\sigma, \omega) \in \Sigma \times \Omega,$$

where ψ is uniform measure on Σ, $\phi_p = \phi_{p,1}$ is product measure on Ω with density p, and 1_F is the indicator function of the event that σ is constant on each open cluster of ω, that is,

$$F = \big\{(\sigma, \omega) \in \Sigma \times \Omega : \sigma_x = \sigma_y \text{ for every } e = \langle x, y\rangle \text{ satisfying } \omega(e) = 1\big\}.$$

The measure μ may be viewed as the product measure $\psi \times \phi_p$ conditioned on the event F. The *marginal* of a measure on a product space is its projection onto a component.

Theorem 1.3.4. *Let $q \in \{2, 3, \dots\}$ and $p \in (0, 1)$.*

(a) Marginal on Σ. *The first marginal of μ (on Σ) is the Potts measure $\pi_{\beta,q}$ where $p = 1 - e^{-\beta}$.*

(b) Marginal on Ω. *The second marginal of μ (on Ω) is the random-cluster measure $\phi_{p,q}$.*

(c) Conditional measures.

  (i) *Given $\omega \in \Omega$, the conditional measure on Σ is obtained by putting (uniformly) random spins on entire clusters of ω. These spins are constant on given clusters, and are independent between clusters.*

(ii) *Given $\sigma \in \Sigma$, the conditional measure on Ω is obtained as follows. For $e = \langle x, y\rangle \in E$, we set $\omega(e) = 0$ if $\sigma_x \neq \sigma_y$, and otherwise $\omega(e) = 1$ with probability p (independently of other edges).*

(c) Partition functions. *We have that*

$$Z_{\mathrm{RC}}(G; p, q) = e^{-\beta|E|} Z_{\mathrm{P}}(G; \beta, q). \tag{1.3}$$

This coupling may be used to show that correlations in Potts models correspond to connection probabilities in random-cluster models (see, for example, [23, Thm 1.16]). In this way, one may harness methods of stochastic geometry in order to understand the correlation structure of the Potts system. The basic theorem of this type is Theorem 1.3.5, following.

The 'two-point correlation function' of the Potts measure $\pi_{\beta,q}$ on the finite graph $G = (V, E)$ is the function

$$\tau_{\beta,q}(x, y) := \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q}, \qquad x, y \in V. \tag{1.4}$$

Let $\{x \leftrightarrow y\}$ be the event of Ω on which there exists an open path joining vertex x to vertex y. The 'two-point connectivity function' of the random-cluster measure $\phi_{p,q}$ is the function $\phi_{p,q}(x \leftrightarrow y)$ for $x, y \in V$, that is, the probability that x and y are joined by a path of open edges. It turns out that these two-point functions are the same up to a constant factor.

Theorem 1.3.5 (Correlation/connection theorem)**.** *If $q \in \{2, 3, \dots\}$ and $p = 1 - e^{-\beta} \in (0, 1)$, then*

$$\tau_{\beta,q}(x, y) = (1 - q^{-1})\phi_{p,q}(x \leftrightarrow y), \qquad x, y \in V. \tag{1.5}$$

The Potts models considered above have zero external field. Some complications arise when an external field is added; see Section 1.3.5.

1.3.4 Partition functions and the Tutte polynomial

The Potts and random-cluster partition functions may be viewed as evaluations of the Tutte polynomial, as follows.

Theorem 1.3.6. *Let $p \in (0, 1)$, $q \in (0, \infty)$, and*

$$u - 1 = \frac{q(1-p)}{p}, \qquad v - 1 = \frac{p}{1-p}.$$

(a) *The partition function $Z_{\mathrm{RC}}(G)$ of the random-cluster measure on G with parameters p, q satisfies*

$$Z_{\mathrm{RC}}(G) = \frac{(u-1)(v-1)^{|V|}}{v^{|E|}} T(G; u-1, v-1).$$

(b) *If $q \in \{2, 3, \dots\}$ and $p = 1 - e^{-\beta}$, the partition function of the q-state Potts model on G satisfies*

$$Z_{\mathrm{P}}(G; \beta, q) = (u-1)(v-1)^{|V|} T(G; u-1, v-1).$$

1.3.5 Potts extensions

There are three senses in which the Potts model of Definition 1.3.2 may be said to be in its simplest form: (i) each edge plays an equal (deterministic) role, (ii) the external field satisfies $h = 0$, and (iii) the model is *ferromagnetic*. More generally, one may consider the partition function $Z_{\mathrm{P}}(G; \beta, \mathbf{J}, h) = \sum_{\sigma \in \Sigma} e^{-\beta H_{\mathrm{P}}(\sigma)}$ where the Hamiltonian is given by

$$H_{\mathrm{P}}(\sigma) = -\sum_{e=\langle x,y \rangle} J_e \delta_{\sigma_x, \sigma_y} - \sum_{j=1}^{q} \sum_{x \in V} h_j \delta_{\sigma_x, j}. \tag{1.6}$$

Here, $\mathbf{J} = (J_e : e \in E)$ is a family of edge-parameters assumed to satisfy $J_e \neq 0$, and $\mathbf{h} = (h_j : j = 1, 2, \dots, q)$ is a vector of external fields. The model is termed *ferromagnetic* if $J_e > 0$ for $e \in E$, and *purely antiferromagnetic* if $J_e < 0$ for all $e \in E$. In the ferromagnetic case, the measure has a property of positive association (as in Section 1.5.2) which is absent in the non-ferromagnetic case. The general Potts partition function of (1.6) poses some new difficulties.

Assume first that $\mathbf{h} = \mathbf{0}$. The associated random-cluster formula yields a function $\phi_{\mathbf{p},q}$ where $p_e = 1 - e^{-\beta J_e}$. If $J_e < 0$ for some e, this does not define a probability measure. In addition, the Potts model does not satisfy the range of correlation inequalities that hold in the ferromagnetic case. On the other hand, Theorem 1.3.6(b) is easily extended for general $\mathbf{J}$ to a multivariate Tutte polynomial on $G = (V, E)$ which may be written in the form

$$Z(G; q, \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e, \tag{1.7}$$

where q and $\mathbf{v} = (v_e : e \in E)$ are viewed as parameters. See [14, 42] for recent accounts.

When $\mathbf{h} \neq \mathbf{0}$, a form of the Tutte–Potts correspondence may be found in [8], where positive association and infinite-volume limits are explored, and also in a slightly more general setting in [14] (see also [34]). It turns out that the Potts partition function Z_{P} arising from (1.6) equals an evaluation of the V-polynomial, namely $V(G, \mathbf{h}; s, \mathbf{p})$ where $s = \sum_j e^{\beta h_j}$, and $\mathbf{p} = (e^{\beta J_e} - 1 : e \in E)$. See [14].

The J_e may themselves be random, in which case the model is termed *quenched*, in contrast to the *annealed* case in which one averages initially over the J_e. If the probability distribution of the J_e allocates strictly positive probability to both positive and negative values, the system is a ‘spin glass’. See [33].

1.4 Phase Transition

Statistical mechanics is based around the notion of *phase transition*. Suppose for simplicity that a given physical system has a single parameter denoted T and called 'temperature'. In many cases in nature, there exists a 'critical temperature', denoted T_c, and the macroscopic behaviour of the system depends on whether $T < T_c$ or $T > T_c$. For example, it was observed by Pouillet [30, 37] and, later, Curie [13] that there exists a threshold temperature T_c for the retention of magnetization by an iron body. This discovery motivated Lenz's proposal of the Ising model of Definition 1.3.1, restricted initially to the case of one dimension and extended subsequently to higher dimensions by Peierls [35] and others.

Within the context of such a mathematical model, a singularity can occur only in an infinite system. The procedure is as follows. The configuration space in question is determined inside a space of size n, say. To each configuration σ is allocated an energy, or 'Hamiltonian', $H(\sigma)$, leading to the 'weight' $w(\sigma) := e^{-\beta H(\sigma)}$, where $\beta = 1/T$ as before. The 'partition function' $Z_n := \sum_\sigma w(\sigma)$ is a smooth function of T and of any other parameters. The 'infinite-volume partition function' is given in the so-called 'thermodynamic limit' by

$$\log Z := \lim_{n\to\infty} \left\{ \frac{1}{n} \log Z_n \right\}.$$

Now, Z is not generally a smooth function, and it is through studying the singularities of Z and its partial derivatives that one obtains a picture of any phase transition. See [39].

An explicit example of the thermodynamic limit and the infinite-volume partition function is exhibited in Theorem 1.9.1.

1.5 Basic Properties of Random-Cluster Measures

This section includes some of the basic properties of a random-cluster measure on the finite graph $G = (V, E)$. Each may be expressed in terms of the Tutte polynomial.

1.5.1 Stochastic ordering

The state space $\Omega = \{0, 1\}^E$ is a partially ordered set with partial order given by: $\omega_1 \le \omega_2$ if $\omega_1(e) \le \omega_2(e)$ for all $e \in E$. This partial order is extremely useful in the analysis of Potts and random-cluster models, and it induces partial orderings on the spaces of associated functions and measures.

Definition 1.5.1.

(a) A random variable $f : \Omega \to \mathbb{R}$ is called *increasing* if $f(\omega_1) \le f(\omega_2)$ whenever $\omega_1 \le \omega_2$.

(b) An event $A \subseteq \Omega$ is called *increasing* if its indicator function 1_A is increasing.

(c) Given two probability measures μ_1, μ_2 on Ω, we write $\mu_1 \le_{\mathrm{st}} \mu_2$, and say that μ_1 is stochastically smaller than μ_2, if $\mu_1(f) \le \mu_2(f)$ for all increasing random variables f on Ω.

Arguably the most useful approach to stochastic ordering is due to Holley. We obtain the following comparison inequalities as corollaries of Holley's inequality, see [27] and [23, Thm 2.1].

Theorem 1.5.2 (Comparison inequalities)**.** *It is the case that*

$$\phi_{p',q'} \le_{\mathrm{st}} \phi_{p,q} \qquad \text{if} \quad q' \ge q,\ q' \ge 1,\ \text{and } p' \le p,$$

$$\phi_{p',q'} \ge_{\mathrm{st}} \phi_{p,q} \qquad \text{if} \quad q' \ge q,\ q' \ge 1,\ \text{and } \frac{p'}{q'(1-p')} \ge \frac{p}{q(1-p)}.$$

1.5.2 Positive association

Holley's inequality admits a neat proof of the FKG inequality of [21]. This amounts to the following in the case of random-cluster measures.

Theorem 1.5.3 (Positive association)**.** *Let $p \in (0,1)$ and $q \in [1,\infty)$. If f and g are increasing functions on Ω, then*

$$\phi_{p,q}(fg) \ge \phi_{p,q}(f)\phi_{p,q}(g).$$

Specialising to indicator functions $f = 1_A$, $g = 1_B$, we obtain that

$$\phi_{p,q}(A \cap B) \ge \phi_{p,q}(A)\phi_{p,q}(B) \qquad \text{for increasing events } A, B,$$

whenever $q \ge 1$. Positive association is generally false when $0 < q < 1$.

1.6 The Limit as $q \downarrow 0$

1.6.1 UST, USF, and USC

Some interesting limits with combinatorial flavours arise from consideration of $\phi_{p,q}$ as $q \downarrow 0$. Write Ω_{for}, Ω_{st}, Ω_{cs} for the subsets of Ω containing all forests, spanning trees, and connected subgraphs, respectively, and write USF, UST, and UCS for the uniform probability measures on the respective sets Ω_{for}, Ω_{st}, Ω_{cs}. An account of the following limits and their history may be found at [23, Thm 1.2].

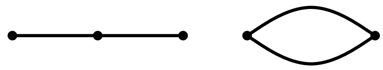

FIGURE 1.2
Two edges in series and in parallel.

Theorem 1.6.1. *We have in the limit as* $q \downarrow 0$ *that*:

$$\phi_{p,q} \Rightarrow \begin{cases} \text{UCS} & \textit{if } p = \frac{1}{2}, \\ \text{UST} & \textit{if } p \to 0 \textit{ and } q/p \to 0, \\ \text{USF} & \textit{if } p = q. \end{cases}$$

The spanning tree limit UST is especially interesting for historical and mathematical reasons. The random-cluster model originated in a systematic study by Fortuin and Kasteleyn of systems of a certain type which satisfy certain parallel and series laws. Electrical networks are the best known such systems: two resistors of resistances r_1 and r_2 in parallel (respectively, in series) may be replaced by a single resistor with resistance $(r_1^{-1} + r_2^{-1})^{-1}$ (respectively, $r_1 + r_2$); see Figure 1.2. Fortuin and Kasteleyn realized that the electrical-network theory of a graph G is related to the limit as $q \downarrow 0$ of the random-cluster model on G, where p is given by $p = \sqrt{q}/(1+\sqrt{q})$. It has been known since Kirchhoff's theorem [31] that the electrical currents which flow in a network may be expressed in terms of counts of spanning trees.

The theory of the uniform spanning tree measure UST is beautiful in its own right. In partnership with the so-called 'loop erased random walk', it is linked in an important way to the emerging field of stochastic growth processes of 'stochastic Löwner evolution' (SLE) type. See Figure 1.3, and the references in [25, Chap. 2].

1.6.2 Negative association

Let E be a finite set, and let μ be a probability measure on the space $\Omega = \{0,1\}^E$. There are several concepts of negative association, of which we present three here.

For $\omega \in \Omega$ and $F \subseteq E$, the cylinder event $\Omega_{F,\omega}$ generated by ω on F is given by

$$\Omega_{F,\omega} = \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for } e \in F\}.$$

For $E' \subseteq E$ and an event $A \subseteq \Omega$, we say that A *is defined on* E' if, for all $\omega \in \Omega$, we have that $\omega \in A$ if and only if $\Omega_{E',\omega} \subseteq A$. Let A and B be events in Ω. We define $A \,\square\, B$ to be the set of all vectors $\omega \in \Omega$ for which there exists $F \subseteq E$ such that $\Omega_{F,\omega} \subseteq A$ and $\Omega_{\overline{F},\omega} \subseteq B$, where $\overline{F} = E \setminus F$. Note that the choice of F is allowed to depend on the vector ω. The operator $\square$ originated in the work of van den Berg and Kesten [7] on the well known BK inequality.

Definition 1.6.2.

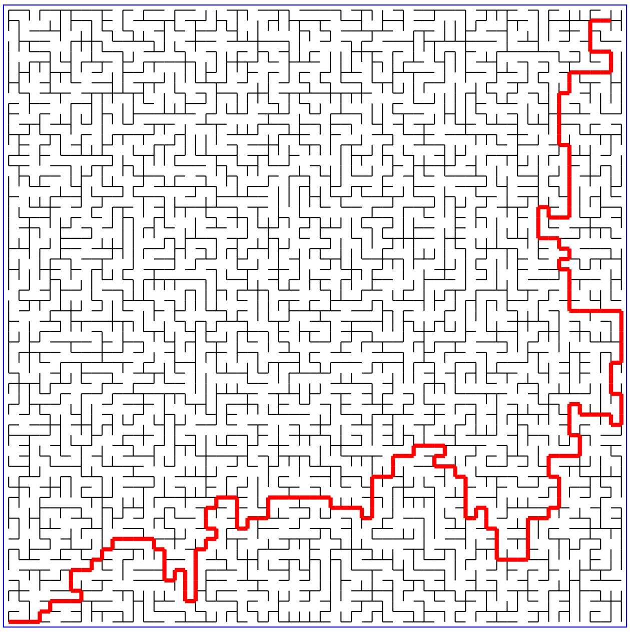

FIGURE 1.3
A uniform spanning tree (UST) on a large box of the square lattice. It contains a unique path between any two vertices, taken here as opposite corners of the box. This path has the law of a loop-erased random walk. (Figure by courtesy of Oded Schramm.)

(a) The measure μ is *edge negatively associated* if

$$\mu(J_e \cap J_f) \le \mu(J_e)\mu(J_f), \qquad e, f \in E,\ e \ne f,$$

where J_e is the event that e is open.

(b) We call μ *negatively associated* if

$$\mu(A \cap B) \le \mu(A)\mu(B)$$

for all pairs (A, B) of increasing events with the property that there exists $E' \subseteq E$ such that A is defined on E' and B is defined on its complement $E \setminus E'$.

(c) We say that μ has the *disjoint occurrence property* if

$$\mu(A \square B) \le \mu(A)\mu(B), \qquad A, B \subseteq \Omega.$$

Proposition 1.6.3. *We have that*

$$\begin{aligned} &\mu \text{ has the disjoint occurrence property} \\ &\qquad \Rightarrow \mu \text{ is negatively associated} \\ &\qquad \Rightarrow \mu \text{ is edge negatively associated.} \end{aligned}$$

The proof of the proposition follows from the definitions. Neither of the two implications of the proposition can be reversed: see [32] for the first, and the second is more elementary.

It was proved by Reimer [38] that the product measures $\phi_{p,1}$ have the disjoint occurrence property. The random-cluster measure $\phi_{p,q}$ cannot (generally) be edge negatively associated when $q > 1$. It may be conjectured that $\phi_{p,q}$ satisfies some form of negative association when $q < 1$. Such a property would be very useful in studying random-cluster measures when $q < 1$.

In the absence of a satisfactory approach to the general case of random-cluster measures with $q < 1$, we turn next to the issue of negative association of $\phi_{p,q}$ in the limit as $q \downarrow 0$.

Conjecture 1.6.4. For any finite graph $G = (V, E)$, the uniform spanning forest measure USF and the uniform connected subgraph measure UCS are edge negatively associated.

A stronger version of this conjecture is that USF and UCS are negatively associated in one or both of the further senses described above. Numerical evidence for the conjecture is found in [26]. (The problem is simpler in the symmetric context of USF on the complete graph, see [43].)

The UST measure is, in contrast, much better understood, owing to the theory of electrical networks and, more particularly, Kirchhoff's matrix–tree theorem, [31] and its ramifications. The following was proved by Feder and Mihail, [17].

Theorem 1.6.5. *The uniform spanning tree measure* UST *is negatively associated.*

In addition, UST has the stronger property of being 'strong Rayleigh', see [12]. The material in this section may be found in expanded form in [23, 25].

1.7 Flow Polynomial

1.7.1 Potts correlations and flow counts

The Potts correlation functions (1.4) may be expressed in terms of flow polynomials associated with a certain Poissonian random graph derived from G by replacing each edge by a random number of copies.

For a vector $\mathbf{m} = (m_e : e \in E)$ of non-negative integers, let $G_{\mathbf{m}} = (V, E_{\mathbf{m}})$ be the graph with vertex set V and, for each $e \in E$, with exactly m_e edges in parallel joining the endvertices of the edge e; the original edge e is removed.

Let $\beta \geq 0$, and let $\mathbf{P} = (P_e : e \in E)$ be a family of independent random variables such that P_e has the Poisson distribution with parameter β. The random graph $G_{\mathbf{P}} = (V, E_{\mathbf{P}})$ is called a *Poisson graph with intensity* β. Let $\mathbb{P}_\beta$ and $\mathbb{E}_\beta$ denote the corresponding probability measure and expectation.

For $x, y \in V$, let $G_{\mathbf{P}}^{x,y}$ denote the graph obtained from $G_{\mathbf{P}}$ by adding an edge with endvertices x, y. If x and y are adjacent in the original graph $G_{\mathbf{P}}$, we add a further edge between them. Potts correlations are related to flow counts as follows.

Theorem 1.7.1. *Let $q \in \{2, 3, \dots\}$ and $\beta \geq 0$. Then*

$$q\tau_{\beta,q}(x,y) = \frac{\mathbb{E}_\beta(F(G_{\mathbf{P}}^{x,y}; q))}{\mathbb{E}_\beta(F(G_{\mathbf{P}}; q))}, \qquad x, y \in V. \tag{1.1}$$

This formula is particularly striking when $q = 2$, since non-zero mod-2 flows take only the value 1. A finite graph $H = (W, F)$ is said to be *even* if every vertex has even degree. Evidently $F(H; 2) = 1$ if H is even, and $F(H; 2) = 0$ otherwise. By (1.1), for any graph G,

$$q\tau_{\beta,q}(x,y) = \frac{\mathbb{P}_\beta(G_{\mathbf{P}}^{x,y} \text{ is even})}{\mathbb{P}_\beta(G_{\mathbf{P}} \text{ is even})}, \tag{1.2}$$

when $q = 2$. This observation is central to the so called 'random-current expansion' of the Ising model, which has proved very powerful in the study of both classical and quantum Ising models. See [1, 2, 3, 4, 9].

Theorem 1.7.1 may be extended via (1.3.5) to the random-cluster model. The following is obtained by expressing the flow polynomial in terms of the Tutte polynomial T, and allowing q to vary continuously.

Theorem 1.7.2. *Let $p \in (0,1)$, $q \in (0, \infty)$, and let β satisfy $p = 1 - e^{-\beta q}$.*

(i) *For $x, y \in V$,*

$$(q-1)\phi_{G,p,q}(x \leftrightarrow y) = \frac{\mathbb{E}_\beta\big((-1)^{1+|E_{\mathbf{P}}|} T(G_{\mathbf{P}}^{x,y}; 0, 1-q)\big)}{\mathbb{E}_\beta\big((-1)^{|E_{\mathbf{P}}|} T(G_{\mathbf{P}}; 0, 1-q)\big)}. \tag{1.3}$$

(ii) *For $q \in \{2, 3, \dots\}$,*

$$Z_{\mathrm{RC}}(G; p, q) = \phi_{G,p}(q^{k(\omega)}) = (1-p)^{|E|(q-2)/q} q^{|V|} \mathbb{E}_\beta(F(G_{\mathbf{P}}; q)). \tag{1.4}$$

Further details may be found in [24].

1.7.2 The random-current expansion when $q = 2$

Unlike the Potts model, there is a fairly complete analysis of the Ising model. A principal part in this analysis is played by Theorem 1.7.1 with $q = 2$ under the heading 'random-current expansion'. This has permitted proofs amongst other things of the exponential decay of correlations in the low-β regime on the cubic lattice $\mathbb{L}^d$ with $d \geq 2$. It has not so far been possible to extend this work to general Potts models, but Theorem 1.7.1 could play a part in such an extension.

Let $G = (V, E)$ be a finite graph and set $q = 2$. By Theorem 1.7.1,

$$2\tau_{\beta,2}(x, y) = \frac{\mathbb{P}_\beta(G_{\mathbf{P}}^{x,y} \text{ is even})}{\mathbb{P}_\beta(G_{\mathbf{P}} \text{ is even})}, \qquad 0 \leq \beta < \infty. \tag{1.5}$$

There is an important correlation inequality known as Simon's inequality, [40]. Let $x, z \in V$ be distinct vertices. A set W of vertices is said to *separate* x and z if $x, z \notin W$ and every path from x to z contains some vertex of W.

Theorem 1.7.3. *Let $x, z \in V$ be distinct vertices, and let W separate x and z. Then $\kappa_{\beta,2}(x, y) := 2\tau_{\beta,2}(x, y)$ satisfies*

$$\kappa_{\beta,2}(x, z) \leq \sum_{y \in W} \kappa_{\beta,2}(x, y)\kappa_{\beta,2}(y, z).$$

The Ising model corresponds to a random-cluster measure $\phi_{p,q}$ with $q = 2$. By (1.5),

$$\kappa_{\beta,q}(x, y) = \phi_{p,q}(x \leftrightarrow y),$$

where $p = 1 - e^{-\beta q}$ and $q = 2$. The Simon inequality may be written in the form

$$\phi_{p,q}(x \leftrightarrow z) \leq \sum_{y \in W} \phi_{p,q}(x \leftrightarrow y)\phi_{p,q}(y \leftrightarrow z) \tag{1.6}$$

whenever W separates x and z. It is well known that this inequality is valid also when $q = 1$, see [22, Chap. 6]. One may conjecture that it holds for any $q \in [1, 2]$.

1.8 The Limit of Zero Temperature

The physical interpretation of the constant β is as $\beta = 1/(kT)$ where k is Boltzmann's constant and T denotes (absolute) temperature. The limit $T \downarrow 0$ corresponds to the limit $\beta \to \infty$. The ferromagnetic Potts measure $\pi_{\beta,q}$ on a finite graph $G = (V, E)$ converges weakly to the probability measure that allocates a uniform random spin to each connected component of G, this being constant on each component and independent between components. A realization of this recipe is called a 'ground state' of the system.

The situation is more interesting in the presence of a vector $\mathbf{J}$ of edge-parameters, some of which are negative. The ground states in this case are colourings κ of V with the colour palette $\{1, 2, \dots, q\}$ and the property that, for any each $e = \langle x, y \rangle$,

$$\kappa(x) \begin{cases} = \kappa(y) & \text{if } J_e > 0, \\ \neq \kappa(y) & \text{if } J_e < 0. \end{cases} \tag{1.1}$$

In the purely antiferromagnetic case, such a colouring κ is has the property that any two neighbours have different colours.

Theorem 1.8.1. *For the purely antiferromagnetic Potts model,*

$$Z_{\mathrm{P}}(G; \beta, q) \to \chi(G; q) \qquad \text{as } \beta \to \infty,$$

where χ denotes the chromatic polynomial.

This is easily seen to hold, since, as $\beta J_e \to -\infty$, there is limiting weight zero on any pair of *equal* adjacent spins. This observation may be extended naturally in the presence of negative external fields (as in (1.6)) to counts of *list colourings* of G in which the available colours at any given $v \in V$ is restricted to a given list (see [15]).

For given q, there exist graphs G for which (1.1) has no solution, and such graphs are called 'frustrated'.

1.9 The Random-Cluster Model on the Complete Graph

When the underlying graph is the complete graph K_n, the asymptotic behaviour of the corresponding random-cluster partition function $Z_{\mathrm{RC}}(n, p, q)$ may be studied using a mixture of combinatorics and probability, within the regime $q \geq 1$, $p = \lambda/n$. Here is some notation and explanation, in preparation for the main theorem.

Let $q \geq 1$ and $p = \lambda/n$. It turns out that there is a critical value of λ that marks the arrival of a giant cluster in the random-cluster model on K_n, and this value is given in [11] by

$$\lambda_{\mathrm{c}}(q) = \begin{cases} q & \text{if } q \in (0, 2], \\ 2\left(\dfrac{q-1}{q-2}\right) \log(q-1) & \text{if } q \in (2, \infty), \end{cases}$$

and plotted in Figure 1.4. As λ increases through the value λ_{c}, a giant cluster of size approximately $\theta(\lambda, q)n$ is created, where

$$\theta(\lambda, q) = \begin{cases} 0 & \text{if } \lambda < \lambda_{\mathrm{c}}(q), \\ \theta_{\max} & \text{if } \lambda \geq \lambda_{\mathrm{c}}(q), \end{cases}$$

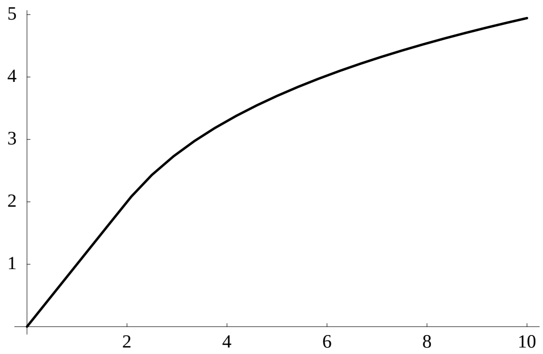

FIGURE 1.4
The critical value $\lambda_c(q)$ plotted against $q > 0$. There is a discontinuity in the second derivative at the value $q = 2$.

and $\theta_{\max}$ is the largest root of the equation

$$e^{-\lambda\theta} = \frac{1-\theta}{1+(q-1)\theta}.$$

Theorem 1.9.1. *Let $q \in (0,\infty)$ and $\lambda \in (0,\infty)$. We have that*

$$\frac{1}{n}\log Z_{\mathrm{RC}}(n,\lambda/n,q) \to \eta(\lambda) \qquad \textit{as } n \to \infty,$$

where

$$\eta(\lambda) = \frac{g(\theta(\lambda))}{2q} - \frac{q-1}{2q}\lambda + \log q,$$
$$g(\theta) = -(q-1)(2-\theta)\log(1-\theta) - \big[2+(q-1)\theta\big]\log\big[1+(q-1)\theta\big].$$

By Theorem 1.3.6, this provides an asymptotic evaluation of the Tutte polynomial $T_{n,\lambda/n,q}(u-1,v-1)$ within the quadrant $u,v \in (0,\infty)$. See [11] and [23, Chap. 10] for further details.

1.10 Open Problems

There is an enormous range of open problems associated with Ising, Potts, and random-cluster models. Of these, there follows a brief selection some of which may be allied in part to the Tutte polynomial.

1. Prove or disprove some version of negative association for the uniform forest measure USF or the uniform connected subgraph measure UCS. See Conjecture 1.6.4.

2. Prove or disprove some version of negative association for the random-cluster measure $\phi_{p,q}$ with $0 < q < 1$.
3. Prove or disprove a version of Simon's inequality for a random-cluster measure $\phi_{p,q}$ with $q \in [1, 2]$, as in (1.6).
4. Establish a version of Simon's inequality, Theorem 1.7.3, for the Potts model with $q \geq 3$.
5. More generally, find an application of mod-q flows to the q-state Potts model with $q \geq 3$, as in Theorem 1.7.1. Develop such an application for real $q > 0$, as in Theorem 1.7.2.
6. There is a universe of problems associated with classical and quantum phase transitions, of which we mention the determination of the natures of the phase transitions of the Potts model with $q \geq 2$ on lattices in dimensions $d \geq 2$.
7. A problem of great current interest to probabilists and mathematical physicists is to understand the geometry of interfaces in the 3-state Potts model on the square lattice $\mathbb{Z}^2$, and more generally the random-cluster model on $\mathbb{Z}^2$ with $1 < q < 4$. See [41], and [6] for a recent reference.

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