Liouville quantum gravity and the Brownian map

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Overview

Part I: Picking surfaces at random
1. Discrete: random planar maps
2. Continuum: Liouville quantum gravity (LQG)
3. Relationship

Part II: The $\text{QLE}(8/3, 0)$ metric on $\sqrt{8/3}\text{-LQG}$
1. First passage percolation on random planar maps
2. First passage percolation on $\sqrt{8/3}\text{-LQG}: \text{QLE}(8/3, 0)$
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[Diagram of a planar map with faces highlighted]
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- First studied by Tutte in 1960s while working on the four color theorem.

**Combinatorics**: enumeration formulas

**Physics**: statistical physics models: percolation, Ising, UST ...

**Probability**: “uniformly random surface,” Brownian surface
Random quadrangulation with 25,000 faces

(Simulation due to J.F. Marckert)
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Brownian map also described in terms of trees (CRT)

(Markert-Mokkadem)
Picking a surface at random in the continuum

**Uniformization theorem:** every Riemannian surface homeomorphic to the unit disk \( D \) can be conformally mapped to the disk.

 Isothermal coordinates: Metric for the surface takes the form 

\[ e^{\rho(z)} \, dz \]

for some smooth function \( \rho \) where \( dz \) is the Euclidean metric.

⇒ Can parameterize the surfaces homeomorphic to \( D \) with smooth functions on \( D \).

▶ If \( \rho = 0 \), get \( D \)

▶ If \( \Delta \rho = 0 \), i.e. if \( \rho \) is harmonic, the surface described is flat

**Question:** Which measure on \( \rho \)? If we want our surface to be a perturbation of a flat metric, natural to choose \( \rho \) as the canonical perturbation of a harmonic function.
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![Diagram](image)

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- Measure on functions $h: D \to \mathbb{R}$ for $D \subseteq \mathbb{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbb{R}^{|D|}$:

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- Continuum GFF not a function — only a generalized function
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  where \( h \) is a GFF and \( \gamma \in [0, 2) \)

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This talk is about endowing each of these objects with the other’s structure and showing they are equivalent.
Canonical embedding of TBM into $S^2$

- TBM is an abstract metric measure space homeomorphic to $S^2$, but it does not obviously come with a canonical embedding into $S^2$. It is believed that there should be a “natural embedding” of TBM into $S^2$ and that the embedded surface is described by a form of Liouville quantum gravity (LQG) with $\gamma = \sqrt{8}/3$. 

Discrete approach: take a uniformly random planar map and embed it conformally into $S^2$ (circle packing, uniformization, etc...), then in the $n \to \infty$ limit it converges to a form of $\sqrt{8}/3$-LQG. Not the approach we will describe today...
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Main result

Theorem (M., Sheffield)

Suppose that \((M, d, \mu)\) is an instance of TBM. Then there exists a Hölder homeomorphism \(\varphi : (M, d) \rightarrow S^2\) such that the pushforward of \(\mu\) by \(\varphi\) has the law of a \(\sqrt{8/3}\)-LQG sphere \((S^2, h)\).
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Comments

1. Construction is purely in the continuum
2. Proof by endowing a metric space structure directly on \(\sqrt{8/3}\)-LQG using the growth process \(\text{QLE}(8/3, 0)\)
3. Resulting metric space structure is shown to satisfy axioms which characterize TBM
4. Separate argument shows the embedding of TBM into \(\sqrt{8/3}\)-LQG is determined by TBM
5. Metric construction is for the \(\sqrt{8/3}\)-LQG sphere. By absolute continuity, can construct a metric on any \(\sqrt{8/3}\)-LQG surface.
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Construction of the metric on $\sqrt{8/3}$-LQG
Detour: first passage percolation (FPP)

- Associate with a graph $(V, E)$ i.i.d. $\text{exp}(1)$ edge weights

Introduced by Eden (1961) and Hammersley and Welsh (1965)

On $\mathbb{Z}^2$?

Question: Large scale behavior of shape of ball wrt perturbed metric?

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Computer simulations show that it is not a Euclidean disk

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Computer simulations show that it is not a Euclidean disk.

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FPP on random planar maps I

- RPM, random vertex $x$. Perform FPP from $x$ (Angel's peeling process).

Important observations:
- Conditional law of map given growth at time $n$ only depends on the boundary lengths of the outside components.
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First passage percolation on random planar maps II

**Goal:** Make sense of FPP in the continuum on top of a LQG surface

- We do not know how to take a continuum limit of FPP on a random planar map and couple it directly with LQG
- Explain a discrete variant of FPP that involves two operations that we do know how to perform in the continuum:
  - Sample random points according to boundary length
  - Draw (scaling limits of) critical percolation interfaces ($\text{SLE}_6$)
FPP on random planar maps II

**Variant:**
- Pick two edges on outer boundary of cluster
FPP on random planar maps II

**Variant:**

- Pick two *edges* on outer boundary of cluster
- Color vertices between edges blue and yellow

This exploration also respects the Markovian structure of the map.

Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
FPP on random planar maps II

**Variant:**

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
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- Pick two edges on outer boundary of cluster
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Continuum limit ansatz

- Sample a random planar map
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $\frac{1}{2}$ and draw percolation interface
- Conformally map to the sphere
- Ansatz: Image of random map converges to a $\sqrt{8}/3$-LQG surface and the image of the interface converges to an independent SLE$_6$. 

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$(\sqrt{8}/3$-LQG surface and the image of the interface converges to an independent $\text{SLE}_6$.)
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**Ansatz** Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE$_6$. 
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
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- Start off with $\sqrt{8/3}$-LQG surface
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Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $\times$
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- Repeat

$\text{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion. $\text{QLE}(8/3, 0)$ is SLE$_6$ with tip re-randomization.
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- Repeat
- Know the conditional law of the LQG surface at each stage

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Discrete approximation of QLE(8/3, 0). Metric ball on a $\sqrt{8/3}$-LQG.
Emergence of TBM in $\sqrt{8/3}$-LQG

- So far, have described a growth process $\text{QLE}(8/3, 0)$ which is a candidate for growth of a metric ball on $\sqrt{8/3}$-LQG.
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- Not obvious that $\text{QLE}(8/3, 0)$ corresponds to the metric balls in a metric space.
- Requires an additional argument — make use of a trick developed by Sheffield, Watson, Wu in the context of $\text{CLE}_4$. Reduces (in a non-trivial way) to the reversibility of whole-plane $\text{SLE}_6$. 

Still a lot of work to show that resulting metric space structure has the law of TBM and that $\sqrt{8/3}$-LQG and TBM are measurable with respect to each other. But can start to see the Brownian map structure emerge: boundary lengths of metric balls in both spaces evolve in the same way.
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Quantum Loewner evolution

QLE(8/3, 0) is a member of a family of processes which are candidates for the scaling limits of DLA and the dielectric breakdown model on LQG surfaces.

More in Scott Sheffield’s talk on Friday.
Further questions

- What is the law of the geodesics for $\sqrt{8/3}$-LQG?
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Thanks!