Liouville Quantum Gravity as a Mating of Trees

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Overview

Part I: Gluing a pair of CRTs

Part II: Scaling limits of random planar maps and Liouville quantum gravity

Part III: Results
Part I: Gluing a pair of CRTs
Gluing a pair of CRTs

$X, Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

Identify points on the graph of $X$ if they are connected by a horizontal line which is below the graph; yields a continuum random tree (CRT)

Same for $C - Y$ yields an independent CRT

Glue the CRTs together by declaring points on the vertical lines to be equivalent

Q: What is the resulting structure?
A: Sphere with a space-filling path.

Apeanosphere
Gluing a pair of CRTs

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$C - Y_t$

$X_t$

$\uparrow$

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**Q:** What is the resulting structure?

**A:** Sphere with a space-filling path. **Apérmansphere**.
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**Q:** What is the resulting structure?  **A:** Sphere with a space-filling path. A peanosphere.
How to check this?

Theorem (Moore 1925)

Let \( \cong \) be any topologically closed equivalence relation on the sphere \( S^2 \). Assume that each equivalence class is connected and not equal to all of \( S^2 \). Then the quotient space \( S^2 / \cong \) is homeomorphic to \( S^2 \) if and only if no equivalence class separates the sphere into two or more connected components.

- An equivalence relation is topologically closed iff for any two sequences \( (x_n) \) and \( (y_n) \) with
  - \( x_n \cong y_n \) for all \( n \)
  - \( x_n \to x \) and \( y_n \to y \)
- we have that \( x \cong y \).
Constructing a sphere from a pair of trees

- $X, Y$ ind. Brownian excursions on $[0, 1]$
- Red/green lines give an $\cong$-relation on $S^2$

$C - Y_t$

$X_t$

$t$
Constructing a sphere from a pair of trees

- $X, Y$ ind. Brownian excursions on $[0, 1]$
- **Red/green** lines give an $\cong$-relation on $S^2$
- Types of equivalence classes:

![Diagram of a sphere constructed from trees with red and green lines]

$\cong$ is topologically closed and does not separate $S^2$ into two or more components, thus $S^2/\cong$ is homeomorphic to $S^2$.

Following the $V$ lines from left to right gives a space-filling path on $S^2/\cong$.

The sphere/space-filling path pair is a peanoshere $Q$.

What is the canonical embedding of this peanoshere into the Euclidean sphere $S^2$?
Constructing a sphere from a pair of trees

- $X$, $Y$ ind. Brownian excursions on $[0, 1]$
- **Red/green** lines give an $\cong$-relation on $S^2$
- Types of equivalence classes:
  1. Outer boundary of rectangle

$C - Y_t$

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$H =$ horizontal, $V =$ vertical

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  2. V line which does not share an endpoint with a H line

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\[ C - Y_t \]

\[ X_t \]

$\forall t$
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- Following the $V$ lines from left to right gives a space-filling path on $S^2/\sim$

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The sphere/space-filling path pair is a **peanoshere**
Constructing a sphere from a pair of trees

- $X, Y$ ind. Brownian excursions on $[0, 1]$
- Red/green lines give an $\simeq$-relation on $S^2$
- Types of equivalence classes:
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  2. $V$ line which does not share an endpoint with a $H$ line
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- $\simeq$ is topologically closed and does not separate $S^2$ into two or more components, thus $S^2/\simeq$ is homeomorphic to $S^2$
- Following the $V$ lines from left to right gives a space-filling path on $S^2/\simeq$

The sphere/space-filling path pair is a peanosphere

Q: What is the canonical embedding of this peanosphere into the Euclidean sphere $S^2$?
Part II: Scaling limits of random planar maps and Liouville quantum gravity
Random planar maps

- A **planar map** is a finite graph together with an embedding in the plane so that no edges cross.
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- Its **faces** are the connected components of the complement of edges.
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- A map is a quadrangulation if each face has 4 adjacent edges.
Random planar maps

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- Its **faces** are the connected components of the complement of edges.
- A map is a **quadrangulation** if each face has 4 adjacent edges.
- Interested in **random quadrangulations** with \( n \) faces — **random planar map** (RPM).

First studied by Tutte in 1960s while working on the four color theorem.

**Combinatorics**: enumeration formulas

**Physics**: statistical physics models: percolation, Ising, UST ...

**Probability**: “uniformly random surface,” Brownian surface

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- Probability: “uniformly random surface,” Brownian surface
Random quadrangulation with 25,000 faces

(Simulation due to J.F. Marckert)
Laws on quadrangulations

- Natural laws on quadrangulations with $n$ faces.
Laws on quadrangulations

- Natural laws on quadrangulations with \( n \) faces.
  - Uniform measure

Sheffield's Hamburger-Cheeseburger (H-C) bijection encodes an FK-weighted planar map by describing the pair of contour functions which correspond to the tree/dual tree pair.
Laws on quadrangulations

- Natural laws on quadrangulations with $n$ faces.
  - Uniform measure
  - Weighted by the partition function of the FK model with $q \in (0, 4)$:
    - For a fixed quadrangulation $M$, the probability of picking it is proportional to $Z_M = \sum_L q^{\#L/2}$ where the sum is over loop configurations $L$ and $\#L$ is the number of loops in $L$. 

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- Natural to pick a map(loop-configuration) pair \((M, L)\) in the FK weighted case
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- Natural to pick a map/loop-configuration pair \((M, L)\) in the FK weighted case
- Can encode the loops in terms of a tree/dual tree pair
  - Generate the tree by first picking a root
  - Generate the branch from the root to any vertex by following the boundaries of the loop configuration until the vertex is cut off from the root, at which point you branch towards the vertex and continue

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Random quadrangulation

Sampled using H-C bijection.
Red tree

Sampled using H-C bijection.
Red and blue trees

Sampled using H-C bijection.
Path snaking between the trees. Encodes the trees and how they are glued together.

Sampled using H-C bijection.
How was the graph embedded into $\mathbb{R}^2$?

Sampled using H-C bijection.
Can subdivide each quadrilateral to obtain a triangulation without multiple edges.

Sampled using H-C bijection.
Circle pack the resulting triangulation.

Sampled using H-C bijection. Packed with Stephenson's CirclePack.
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What is the “limit” of this embedding? Circle packings are related to conformal maps.

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Random planar map convergence results

**Uniformly random**

- Diameter is \( \asymp n^{1/4} \), profile of distances from random point (Chaissang-Schaefer)

- Existence of subsequential limits after rescaling distances by \( n^{-1/4} \) (Le Gall)

- Existence of limit to the Brownian map (Le Gall, Miermont)

- FK-weighted

  - H-C bijection encodes an FK weighted random planar map in terms of a pair of random discrete trees glued together along a space-filling path

  - Sheffield proved that the contour functions of these two discrete trees properly rescaled converge to a pair of Brownian excursions

- For UST weighted random planar maps (\( q = 0 \)), the CRTs are independent. For general \( q \in (0,4) \), the CRTs are correlated

- Canonical embedding of peanospheres that come from gluing correlated CRTs is thus related to the problem of describing the scaling limits of FK weighted random planar maps embedded into \( \mathbb{C} \cup \{\infty\} \) (Duplantier, Miller, Sheffield)
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Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} \, dz$
  where $h$ is a GFF and $\gamma \in [0, 2)$

\[ \gamma = 0.5 \]

(Number of subdivisions)
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  - Can compute areas of regions and lengths of curves

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\[ \gamma = 2.0 \]
(Number of subdivisions)
Scaling limit conjectures

(Simulation due to J.-F. Marckert)

- Uniform RPM conformally embedded into $S^2$ converges to $\sqrt{8/3}$-LQG as $n \to \infty$
Scaling limit conjectures

- Uniform RPM conformally embedded into $S^2$ converges to $\sqrt{8/3}$-LQG as $n \to \infty$.
- For $q \in [0, 4)$, FK weighted RPM together with loop configuration conformally embedded into $S^2$ converges to $\gamma$-LQG as $n \to \infty$ decorated by an independent CLE$_{\kappa'}$ where

\[
q = 2 + 2 \cos \frac{8\pi}{\kappa'}, \quad \gamma = \sqrt{16/\kappa'} \in [\sqrt{2}, 2), \quad \kappa' \in (4, 8].
\]

(Simulation due to J.-F. Marckert)
Part III: Results
Main result

Theorem (Duplantier, M., Sheffield)

For each $\gamma \in (0, 2)$ there is a type of $\gamma$-LQG surface such that the following are true:

- If we explore with an independent space-filling SLE$_{\kappa'}$ process, $\kappa' = \frac{16}{\gamma^2}$, then the LQG lengths of its left and right sides evolve as a 2D Brownian motion $(L, R)$.
- $(L, R)$ almost surely determine both the $\gamma$-LQG surface and the SLE$_{\kappa'}$.

Comments

- Space-filling SLE$_{\kappa'}$ is the peano curve associated with the continuum tree/dual tree pair which encodes CLE$_{\kappa'}$.
- Combined with the convergence for the H-C bijection, this says that FK weighted RPM converge to CLE$_{\kappa'}$-decorated LQG with respect to the topology where two loop-decorated surfaces are close if the contour functions of their tree/dual tree pair are close.
- For planar lattices, the FK models which have been shown to converge to SLE are the UST ($q = 0$), percolation ($q = 1$), FK-Ising model ($q = 2$) (Lawler-Schramm-Werner, Smirnov).
- The above result implies the convergence for all $q \in [0, 4)$ on RPM to SLE$_{\kappa'}$ with $q = 2 + 2 \cos \frac{8\pi}{\kappa'}$, $\gamma = \sqrt{\frac{16}{\kappa'}} \in \left[\sqrt{2}, 2\right)$, $\kappa' \in (4, 8]$. 
- As in the discrete setting, the contour functions of the continuum tree/dual tree pair determine everything.

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Main result

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For each $\gamma \in (0, 2)$ there is a type of $\gamma$-LQG surface such that the following are true:

- If we explore with an independent space-filling $SLE_{\kappa'}$ process, $\kappa' = \frac{16}{\gamma^2}$, then the LQG lengths of its left and right sides evolve as a 2D Brownian motion $(L, R)$
- $(L, R)$ almost surely determine both the $\gamma$-LQG surface and the $SLE_{\kappa'}$
Main result

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For each $\gamma \in (0, 2)$ there is a type of $\gamma$-LQG surface such that the following are true:

- If we explore with an independent space-filling SLE$_{\kappa'}$ process, $\kappa' = \frac{16}{\gamma^2}$, then the LQG lengths of its left and right sides evolve as a 2D Brownian motion $(L, R)$
- $(L, R)$ almost surely determine both the $\gamma$-LQG surface and the SLE$_{\kappa'}$

Comments
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- Combined with the convergence for the H-C bijection, this says that FK weighted RPM converge to $\text{CLE}$-decorated LQG with respect to the topology where two loop-decorated surfaces are close if the contour functions of their tree/dual tree pair are close.
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- The above result implies the convergence for all $q \in [0,4)$ on RPM to $\text{SLE}_{\kappa'}$ with

$$q = 2 + 2 \cos \frac{8\pi}{\kappa'}, \quad \gamma = \sqrt{16/\kappa'} \in [\sqrt{2},2), \quad \kappa' \in (4,8].$$
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- As in the discrete setting, the contour functions of the continuum tree/dual tree pair determine everything
Random quadrangulation as a gluing of trees
Continuum space-filling path

Space-filling $\text{SLE}_6$ on a LQG surface. Random path which encodes the limit of a RPM.
A calculus of random surfaces

- **Types of surfaces:** quantum wedges, cones, disks, and spheres
- **Operations:** welding and cutting
- Interfaces between welded surfaces are variants of SLE which can be described as GFF flow lines
- Conversely, natural to cut these surfaces with SLE-type paths
External inputs

**Imaginary geometry:** calculus of flow lines of $e^{ih/\chi}$ where $h$ is a GFF.

Paths are types of SLE curves. Regions between paths are independent wedges.
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**Conformal welding:** Certain special case of “quantum wedge welding” due to Sheffield. Interface almost surely determined by welding, lengths on left and right sides of interface almost surely agree.
Types of random surfaces

**Quantum wedges**

- Start with a free boundary GFF $h$ on a Euclidean wedge $W_\theta$ with angle $\theta$
Types of random surfaces

Quantum wedges

- Start with a free boundary GFF $h$ on a Euclidean wedge $W_\theta$ with angle $\theta$
- Change coordinates to $H$ with $z^{\theta/\pi}$. Yields free boundary GFF plus $Q(\frac{\theta}{\pi} - 1) \log |z|$

Quantum disks and spheres (finite volume surfaces)

- Constructed with free boundary GFF and Bessel excursion measures

Duplantier, Miller, Sheffield

Liouville Quantum Gravity as a Mating of Trees

September 30, 2014
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- Defined modulo global additive constant; fix additive constant in canonical way
- Parameterize space of wedges by multiple $\alpha$ of $-\log |z|$ or by weight $W = \gamma(\gamma + \frac{2}{\gamma} - \alpha)$

\[\psi(z) = z^{\theta/\pi}\]
\[W_\theta \quad h \quad H\]
\[h \circ \psi + Q \log |\psi'|\]
Types of random surfaces

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Quantum cones

- Similar to a wedge except start with a GFF on a Euclidean cone with angle $\theta$.
- Parameterize space of cones with multiple $\alpha$ of $-\log |z|$ or by weight $W = 2\gamma(Q - \alpha)$. 

$h \circ \psi + Q \log |\psi'|$

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$\psi(z) = z^{\theta/\pi}$

$W_\theta$

$H$

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Types of random surfaces

Quantum wedges

- Start with a free boundary GFF \( h \) on a Euclidean wedge \( W_\theta \) with angle \( \theta \)
- Change coordinates to \( H \) with \( z^{\theta/\pi} \). Yields free boundary GFF plus \( Q(\frac{\theta}{\pi} - 1) \log |z| \)
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Quantum disks and spheres (finite volume surfaces)

- Constructed with free boundary GFF and Bessel excursion measures
Welding and slicing independent wedges

Can “weld” and “slice” quantum wedges to obtain larger/smaller wedges.

- Weight parameter $W = \gamma(\gamma + \frac{2}{\gamma} - \alpha)$ is additive under the welding operation.
- Interface between welding of independent wedges $\mathcal{W}_1, \mathcal{W}_2$ of weight $W_1$ and $W_2$ is an $SLE_\kappa(W_1 - 2; W_2 - 2)$.
- Interface is a deterministic function of $\mathcal{W}_1, \mathcal{W}_2$. 
Welding many wedges

Can also weld together many wedges $\mathcal{W}_1, \ldots, \mathcal{W}_n$ of weight $W_1, \ldots, W_n$ to obtain a wedge $\mathcal{W}$ with weight $W_1 + \cdots + W_n$.

Interfaces are $\text{SLE}_\kappa(\rho_1; \rho_2)$ type processes coupled together as flow lines of a GFF and are a deterministic function of $\mathcal{W}_1, \ldots, \mathcal{W}_n$. 
Welding a wedge to itself

Can “weld” left and right sides of a wedge to obtain a cone. Conversely, can slice a cone with an independent SLE to obtain a wedge.

- Weight parameter $W = 2\gamma(Q - \alpha)$
- Welding left and right sides of weight $W$ wedge yields a weight $W$ cone; the interface is an independent whole-plane $\text{SLE}_\kappa(W - 2)$
- Interface is simple if the wedge is “thick” as on the left (homeomorphic to $\mathbb{H}$); it is self-intersecting if the wedge is thin as on the right (not homeomorphic to $\mathbb{H}$)
Exploring an LQG surface with an $\text{SLE}_{\kappa'}$ with $\kappa' \in (4, 8)$

- Draw an independent $\text{SLE}_{\kappa'}$ on top of a $\frac{3\gamma^2}{2} - 2$ wedge, $\gamma = 4/\sqrt{\kappa'}$
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- Quantum disks cut out by the path have a Poissonian structure
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Gluing independent Lévy trees

Can view $\text{SLE}_{\kappa'}$ process, $\kappa' \in (4, 8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.

The two trees of quantum disks almost surely determine both the $\text{SLE}_{\kappa'}$ and the LQG surface on which it is drawn.

Can convert questions about $\text{SLE}_{\kappa'}$ into questions about $\kappa'_{\frac{\kappa'}{4}}$-stable processes.

Question: Is the graph of components of an $\text{SLE}_{\kappa'}$ process connected?

Equivalently: If we glue together two independent $\kappa'_{\frac{\kappa'}{4}}$-stable trees as above, is it possible to get from one jump to any other by passing through a finite number of $\sim$-classes?
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$C - Y_t$

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Discrete intuition

Welding/cutting results may seem to be a bizarre coincidence at first sight. However, results of this type are very natural in view of conjectures connecting LQG and random planar maps.

“Domain Markov half planar” map with marked boundary edge. Vertices to the left and right of edge colored red and blue.
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Our results in the continuum are analogies of these discrete observations
KPZ interpretation

Can give mathematical treatment of the heuristics used by Duplantier and others to predict quantum and Euclidean dimensions of random fractals.
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- Can deduce quantum scaling exponent; applying the KPZ formula gives Euclidean scaling exponent. Matches rigorously determined value by M., Wu.
Connection with Quantum Loewner Evolution

- Have described two senses in which one can try to show that FK weighted RPM converge to LQG:
  - Conformal embedding
  - Mating of trees

Also natural to show that FK weighted RPM converge to LQG as metric spaces

So far, the metric space limit has only been constructed for uniform RPM ($q=1$): the Brownian map

We have constructed a new universal family of growth processes called QLE (candidate for the scaling limit of DLA, Eden model, and related models on RPM)

We have also recently announced a program to show that QLE ($8/3, 0$) can be used to endow $\sqrt{8/3}$-LQG with a metric which is isometric to the Brownian map

Many steps of this program have already been carried out in the "mating of trees" by Duplantier, Miller, Sheffield
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Thanks!