**STATISTICS** Part IB
**Example Sheet 1 (of 3)** Lent 2022

1. Ask your supervisor to test you on the sheet of common distributions on the course website.

2. If $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ are independent, derive the distribution of $\min(X, Y)$. If $X \sim \Gamma(\alpha, \lambda)$ and $Y \sim \Gamma(\beta, \lambda)$ are independent, derive the distributions of $X + Y$ and $X/(X + Y)$.

3. (a) Let $X_1, \ldots, X_n$ be independent Poisson random variables with $X_i$ having parameter $i\theta$ for some $\theta > 0$. Find a real-valued sufficient statistic $T$, and compute its distribution. Show that the maximum likelihood estimator $\hat{\theta}$ of $\theta$ is unbiased.

   (b) For some $n \geq 2$, let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\theta)$. Find a minimal sufficient statistic $T$, and compute its distribution. Show that the maximum likelihood estimator $\hat{\theta}$ of $\theta$ is biased but asymptotically unbiased. Find an injective function $h$ on $(0, \infty)$ such that, writing $\psi = h(\theta)$, the maximum likelihood estimator $\hat{\psi}$ of the new parameter $\psi$ is unbiased.

4. For some $n \geq 2$ let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Unif}(\theta, 2\theta)$ for some $\theta > 0$. Show that $\tilde{\theta} = 2X_1/3$ is an unbiased estimator of $\theta$. Use the Rao–Blackwell theorem to find an unbiased estimator $\hat{\theta}$ which is a function of a minimal sufficient statistic and which satisfies $\text{Var}(\hat{\theta}) < \text{Var}(\tilde{\theta})$ for all $\theta > 0$.

5. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Unif}(0, \theta)$. Find the maximum likelihood estimator $\hat{\theta}$ of $\theta$. By considering the distribution of $\hat{\theta}/\theta$ and for $\alpha \in (0, 1)$, find an appropriate, one-sided $100(1-\alpha)\%$ confidence interval for $\theta$ based on $\hat{\theta}$.

6. Suppose that $X_1 \sim N(\theta_1, 1)$ and $X_2 \sim N(\theta_2, 1)$ independently, where $\theta_1$ and $\theta_2$ are unknown. Show that both the square $S$ and the circle $C$ in $\mathbb{R}^2$ given by

$$S = \{(\theta_1, \theta_2) \ : \ |\theta_1 - X_1| \leq 2.236, |\theta_2 - X_2| \leq 2.236\}$$
$$C = \{(\theta_1, \theta_2) \ : \ (\theta_1 - X_1)^2 + (\theta_2 - X_2)^2 \leq 5.991\}$$

   are 95% confidence sets for $(\theta_1, \theta_2)$. *Hint:* $\Phi(2.236) = (1 + \sqrt{0.95})/2$ *where* $\Phi$ *is the distribution function of a* $N(0, 1)$ *random variable.* What might be a sensible criterion for choosing between $S$ and $C$?

7. Suppose the number of defects in a silicon wafer can be modelled with a Poisson distribution for which the parameter $\lambda$ is known to be either 1 or 1.5. Suppose the prior mass function for $\lambda$ is

$$\pi_\lambda(1) = 0.4, \qquad \pi_\lambda(1.5) = 0.6.$$

   A random sample of five wafers finds $x = (3, 1, 4, 6, 2)$ defects respectively. Show that the posterior distribution for $\lambda$ given $x$ is

$$\pi_{\lambda|Z}(1 \mid x) = 0.012, \qquad \pi_{\lambda|X}(1.5 \mid x) = 0.988.$$

8. (a) Suppose $X = (X_1, \ldots, X_n)$ has probability density function $f_X(\cdot; \theta)$, and suppose $T$ is a sufficient statistic for $\theta$. Let $\hat{\theta}_{\text{MLE}}$ be the unique maximum likelihood estimator of $\theta$. Show that $\hat{\theta}_{\text{MLE}}$ is a function of $T$.

   (b) Now adopt a Bayesian perspective, and suppose that the parameter $\theta$ has a prior density function $\pi_\theta$. Let the estimator $\hat{\theta}_{\text{Bayes}}$ be the unique minimiser of the expected value of the loss function $L$ under the posterior distribution. Show that $\hat{\theta}_{\text{Bayes}}$ is also a function of $T$.

9. Let $X_1, \ldots, X_n$ be independent and identically distributed with conditional probability density function $f(x \mid \theta) = \theta x^{\theta-1} \mathbb{1}_{\{0 \leq x \leq 1\}}$ for some $\theta > 0$. Suppose the prior distribution for $\theta$ is $\Gamma(\alpha, \lambda)$. Find the posterior distribution of $\theta$ given $X = (X_1, \ldots, X_n)$ and the Bayesian point estimator of $\theta$ under the quadratic loss function.

10. **(Law of small numbers)** For each $n \in \mathbb{N}$, let $X_{n1}, \dots, X_{nn} \overset{\text{iid}}{\sim}$ Bernoulli$(p_n)$ and let $S_n = \sum_{i=1}^n X_{ni}$. Prove that if $np_n \to \lambda \in (0, \infty)$ as $n \to \infty$, then for each $x \in \{0, 1, 2, \dots\}$,

$$\mathbb{P}(S_n = x) \to \mathbb{P}(Y = x)$$

as $n \to \infty$ where $Y \sim$ Poisson$(\lambda)$.

11. For some $n \geq 3$, let $\varepsilon_1, \dots, \varepsilon_n \overset{\text{iid}}{\sim} N(0,1)$, set $X_1 = \varepsilon_1$ and $X_i = \theta X_{i-1} + (1-\theta^2)^{1/2}\varepsilon_i$ for $i = 2, \dots, n$ and some $\theta \in (-1, 1)$. Find a sufficient statistic for $\theta$ that takes values in a subset of $\mathbb{R}^3$.

12. **(Harder)** Let $\hat{\theta}$ be an unbiased estimator of $\theta \in \Theta = \mathbb{R}$ satisfying $\mathbb{E}_\theta(\hat{\theta}^2) < \infty$ for all $\theta \in \Theta$. We say that $\hat{\theta}$ is a uniform minimum variance unbiased (UMVU) estimator if $\text{Var}_\theta(\hat{\theta}) \leq \text{Var}_\theta(\tilde{\theta})$ for all $\theta \in \Theta$ and any other unbiased estimator $\tilde{\theta}$. Prove that a necessary and sufficient condition for $\hat{\theta}$ to be a UMVU estimator is that $\mathbb{E}_\theta(\hat{\theta}U) = 0$ for all $\theta \in \Theta$ and all estimators $U$ with $\mathbb{E}_\theta(U) = 0$ and $\mathbb{E}_\theta(U^2) < \infty$ (i.e. '$\hat{\theta}$ is uncorrelated with every unbiased estimator of 0'). Is the estimator $\hat{\theta}$ in Question 4 a UMVU estimator?