Paper 3, Section I

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with state space S.

(i) What does it mean to say that $(X_n)_{n\geqslant 0}$ has the strong Markov property? Your answer should include the definition of the term *stopping time.*

(ii) Show that

$$\mathbb{P}(X_n = i \text{ at least } k \text{ times } |X_0 = i) = \big[\mathbb{P}(X_n = i \text{ at least once } |X_0 = i)\big]^k$$

for a state $i \in S$. You may use without proof the fact that $(X_n)_{n\geqslant 0}$ has the strong Markov property.

Paper 4, Section I

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain on a state space S, and let $p_{ij}(n) = \mathbb{P}(X_n = j|X_0 = i)$.

(i) What does the term *communicating class* mean in terms of this chain?

(ii) Show that $p_{ii}(m+n) \geqslant p_{ij}(m)p_{ji}(n)$.

(iii) The period d_i of a state i is defined to be

$$d_i = \gcd\{n \geqslant 1 : p_{ii}(n) > 0\}.$$

Show that if i and j are in the same communicating class and $p_{jj}(r) > 0$, then d_i divides r.

Paper 1, Section II

20H Markov Chains

Let $P = (p_{ij})_{i,j \in S}$ be the transition matrix for an irreducible Markov chain on the finite state space S.

(i) What does it mean to say π is the invariant distribution for the chain?

(ii) What does it mean to say the chain is in detailed balance with respect to π?

(iii) A symmetric random walk on a connected finite graph is the Markov chain whose state space is the set of vertices of the graph and whose transition probabilities are

$$p_{ij} = \begin{cases} 1/D_i & \text{if } j \text{ is adjacent to } i \\ 0 & \text{otherwise,} \end{cases}$$

where D_i is the number of vertices adjacent to vertex i. Show that the random walk is in detailed balance with respect to its invariant distribution.

(iv) Let π be the invariant distribution for the transition matrix P, and define an inner product for vectors $x, y \in \mathbb{R}^S$ by the formula

$$\langle x, y \rangle = \sum_{i \in S} x_i \pi_i y_i .$$

Show that the equation

$$\langle x, Py \rangle = \langle Px, y \rangle$$

holds for all vectors $x, y \in \mathbb{R}^S$ if and only if the chain is in detailed balance with respect to π. [Here $z \in \mathbb{R}^S$ means $z = (z_i)_{i \in S}$.]

Paper 2, Section II

20H Markov Chains

(i) Let $(X_n)_{n \geqslant 0}$ be a Markov chain on the finite state space S with transition matrix P. Fix a subset $A \subseteq S$, and let

$$H = \inf\{n \geqslant 0 : X_n \in A\}.$$

Fix a function g on S such that $0 < g(i) \leqslant 1$ for all $i \in S$, and let

$$V_i = \mathbb{E}\left[\prod_{n=0}^{H-1} g(X_n) | X_0 = i\right]$$

where $\prod_{n=0}^{-1} a_n = 1$ by convention. Show that

$$V_i = \begin{cases} 1 & \text{if } i \in A \\ g(i) \sum_{j \in S} P_{ij} V_j & \text{otherwise.} \end{cases}$$

(ii) A flea lives on a polyhedron with N vertices, labelled $1, \ldots, N$. It hops from vertex to vertex in the following manner: if one day it is on vertex $i > 1$, the next day it hops to one of the vertices labelled $1, \ldots, i-1$ with equal probability, and it dies upon reaching vertex 1. Let X_n be the position of the flea on day n. What are the transition probabilities for the Markov chain $(X_n)_{n \geqslant 0}$?

(iii) Let H be the number of days the flea is alive, and let

$$V_i = \mathbb{E}(s^H | X_0 = i)$$

where s is a real number such that $0 < s \leqslant 1$. Show that $V_1 = 1$ and

$$\frac{i}{s} V_{i+1} = V_i + \frac{i-1}{s} V_i$$

for $i \geqslant 1$. Conclude that

$$\mathbb{E}(s^H | X_0 = N) = \prod_{i=1}^{N-1} \left(1 + \frac{s-1}{i}\right).$$

[Hint. Use part (i) with $A = \{1\}$ and a well-chosen function g.]

(iv) Show that

$$\mathbb{E}(H | X_0 = N) = \sum_{i=1}^{N-1} \frac{1}{i}.$$

Paper 3, Section I

9E Markov Chains

An intrepid tourist tries to ascend Springfield's famous infinite staircase on an icy day. When he takes a step with his right foot, he reaches the next stair with probability $1/2$, otherwise he falls down and instantly slides back to the bottom with probability $1/2$. Similarly, when he steps with his left foot, he reaches the next stair with probability $1/3$, or slides to the bottom with probability $2/3$. Assume that he always steps first with his right foot when he is at the bottom, and alternates feet as he ascends. Let X_n be his position after his nth step, so that $X_n = i$ when he is on the stair i, $i = 0, 1, 2, \ldots$, where 0 is the bottom stair.

(a) Specify the transition probabilities p_{ij} for the Markov chain $(X_n)_{n \geqslant 0}$ for any $i, j \geqslant 0$.

(b) Find the equilibrium probabilities π_i, for $i \geqslant 0$. [Hint: $\pi_0 = 5/9$.]

(c) Argue that the chain is irreducible and aperiodic and evaluate the limit

$$\lim_{n \to \infty} \mathbb{P}(X_n = i)$$

for each $i \geqslant 0$.

Paper 4, Section I

9E Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ with state space $\{a, b, c, d\}$ and transition probabilities given by the following table.

	a	b	c	d
a	1/4	1/4	1/2	0
b	0	1/4	0	3/4
c	1/2	0	1/4	1/4
d	0	1/2	0	1/2

By drawing an appropriate diagram, determine the communicating classes of the chain, and classify them as either open or closed. Compute the following transition and hitting probabilities:

- $\mathbb{P}(X_n = b \mid X_0 = d)$ for a fixed $n \geqslant 0$,
- $\mathbb{P}(X_n = c \text{ for some } n \geqslant 1 \mid X_0 = a)$.

Paper 1, Section II

20E Markov Chains

Let $(X_n)_{n \geqslant 0}$ be a Markov chain.

(a) What does it mean to say that a state i is positive recurrent? How is this property related to the equilibrium probability π_i? You do not need to give a full proof, but you should carefully state any theorems you use.

(b) What is a communicating class? Prove that if states i and j are in the same communicating class and i is positive recurrent then j is positive recurrent also.

A frog is in a pond with an infinite number of lily pads, numbered $1, 2, 3, \ldots$. She hops from pad to pad in the following manner: if she happens to be on pad i at a given time, she hops to one of pads $(1, 2, \ldots, i, i+1)$ with equal probability.

(c) Find the equilibrium distribution of the corresponding Markov chain.

(d) Now suppose the frog starts on pad k and stops when she returns to it. Show that the expected number of times the frog hops is $e(k-1)!$ where $e = 2.718\ldots$. What is the expected number of times she will visit the lily pad $k+1$?

Paper 2, Section II

20E Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a simple, symmetric random walk on the integers $\{\dots, -1, 0, 1, \dots\}$, with $X_0 = 0$ and $\mathbb{P}(X_{n+1} = i \pm 1 \,|\, X_n = i) = 1/2$. For each integer $a \geqslant 1$, let $T_a = \inf\{n \geqslant 0 : X_n = a\}$. Show that T_a is a stopping time.

Define a random variable Y_n by the rule

$$Y_n = \begin{cases} X_n & \text{if } n < T_a\,, \\ 2a - X_n & \text{if } n \geqslant T_a\,. \end{cases}$$

Show that $(Y_n)_{n\geqslant 0}$ is also a simple, symmetric random walk.

Let $M_n = \max_{0\leqslant i\leqslant n} X_n$. Explain why $\{M_n \geqslant a\} = \{T_a \leqslant n\}$ for $a \geqslant 0$. By using the process $(Y_n)_{n\geqslant 0}$ constructed above, show that, for $a \geqslant 0$,

$$\mathbb{P}(M_n \geqslant a,\, X_n \leqslant a-1) = \mathbb{P}(X_n \geqslant a+1),$$

and thus

$$\mathbb{P}(M_n \geqslant a) = \mathbb{P}(X_n \geqslant a) + \mathbb{P}(X_n \geqslant a+1).$$

Hence compute

$$\mathbb{P}(M_n = a)$$

when a and n are positive integers with $n \geqslant a$. [Hint: if n is even, then X_n must be even, and if n is odd, then X_n must be odd.]

Paper 3, Section I

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a simple random walk on the integers: the random variables $\xi_n \equiv X_n - X_{n-1}$ are independent, with distribution

$$P(\xi = 1) = p, \quad P(\xi = -1) = q,$$

where $0 < p < 1$, and $q = 1-p$. Consider the hitting time $\tau = \inf\{n : X_n = 0 \text{ or } X_n = N\}$, where $N > 1$ is a given integer. For fixed $s \in (0,1)$ define $\xi_k = E[s^\tau : X_\tau = 0 | X_0 = k]$ for $k = 0, \ldots, N$. Show that the ξ_k satisfy a second-order difference equation, and hence find them.

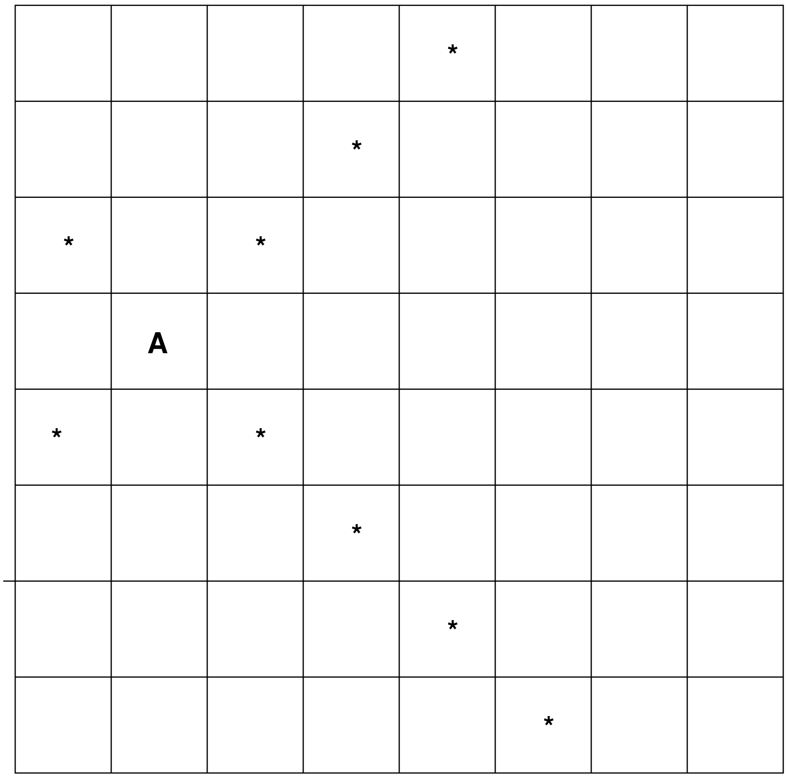
Paper 4, Section I

9H Markov Chains

In chess, a bishop is allowed to move only in straight diagonal lines. Thus if the bishop stands on the square marked A in the diagram, it is able in one move to reach any of the squares marked with an asterisk. Suppose that the bishop moves at random around the chess board, choosing at each move with equal probability from the squares it can reach, the square chosen being independent of all previous choices. The bishop starts at the bottom left-hand corner of the board.

If X_n is the position of the bishop at time n, show that $(X_n)_{n \geqslant 0}$ is a reversible Markov chain, whose statespace you should specify. Find the invariant distribution of this Markov chain.

What is the expected number of moves the bishop will make before first returning to its starting square?

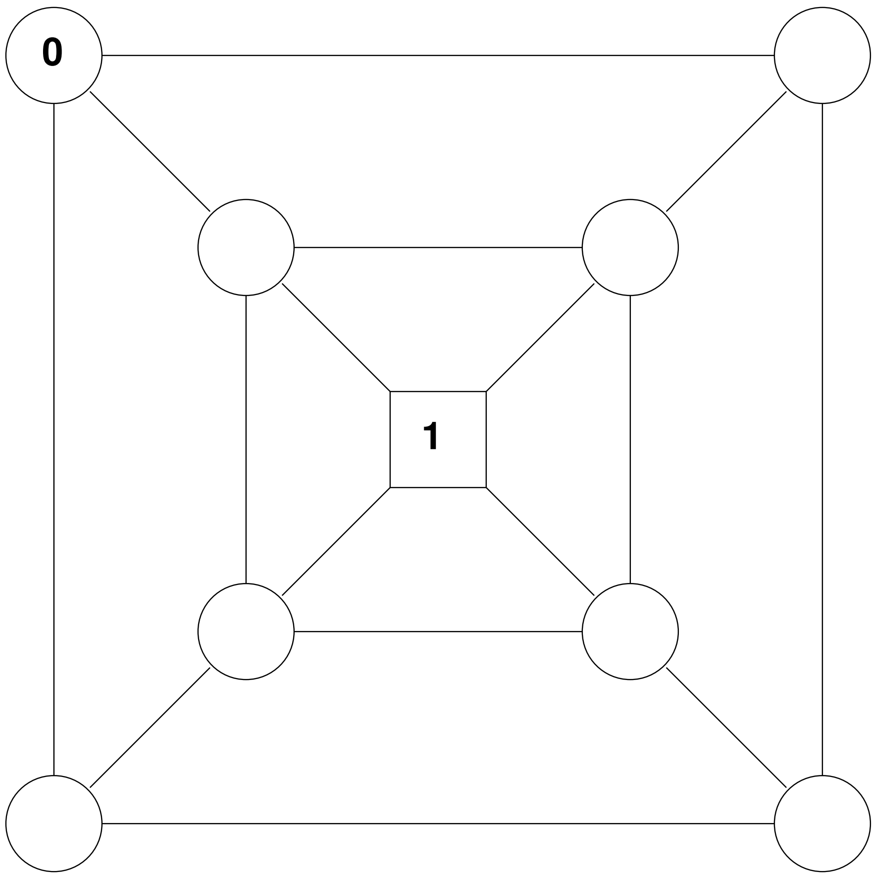


Paper 1, Section II

19H Markov Chains

A gerbil is introduced into a maze at the node labelled 0 in the diagram. It roams at random through the maze until it reaches the node labelled 1. At each vertex, it chooses to move to one of the neighbouring nodes with equal probability, independently of all other choices. Find the mean number of moves required for the gerbil to reach node 1.

Suppose now that the gerbil is intelligent, in that when it reaches a node it will not immediately return to the node from which it has just come, choosing with equal probability from all other neighbouring nodes. Express the movement of the gerbil in terms of a Markov chain whose states and transition probabilities you should specify. Find the mean number of moves until the intelligent gerbil reaches node 1. Compare with your answer to the first part, and comment briefly.



Paper 2, Section II

20H Markov Chains

Suppose that B is a non-empty subset of the statespace I of a Markov chain X with transition matrix P, and let $\tau \equiv \inf\{n \geqslant 0 : X_n \in B\}$, with the convention that $\inf \emptyset = \infty$. If $h_i = P(\tau < \infty | X_0 = i)$, show that the equations

$$(a) \qquad g_i \geqslant (Pg)_i \equiv \sum_{j \in I} p_{ij} g_j \geqslant 0 \quad \forall i,$$

$$(b) \qquad g_i = 1 \quad \forall i \in B$$

are satisfied by $g = h$.

If g satisfies (a), prove that g also satisfies

$$(c) \qquad g_i \geqslant (\tilde{P}g)_i \quad \forall i,$$

where

$$\tilde{p}_{ij} = \begin{cases} p_{ij} & (i \notin B), \\ \delta_{ij} & (i \in B). \end{cases}$$

By interpreting the transition matrix $\tilde{P}$, prove that h is the minimal solution to the equations (a), (b).

Now suppose that P is irreducible. Prove that P is recurrent if and only if the only solutions to (a) are constant functions.

1/II/19H Markov Chains

The village green is ringed by a fence with N fenceposts, labelled $0, 1, \ldots, N-1$. The village idiot is given a pot of paint and a brush, and started at post 0 with instructions to paint all the posts. He paints post 0, and then chooses one of the two nearest neighbours, 1 or $N-1$, with equal probability, moving to the chosen post and painting it. After painting a post, he chooses with equal probability one of the two nearest neighbours, moves there and paints it (regardless of whether it is already painted). Find the distribution of the last post unpainted.

2/II/20H Markov Chains

A Markov chain with state–space $I = \mathbb{Z}^+$ has non-zero transition probabilities $p_{00} = q_0$ and

$$p_{i,i+1} = p_i \,, \quad p_{i+1,i} = q_{i+1} \qquad (i \in I)\,.$$

Prove that this chain is recurrent if and only if

$$\sum_{n \geqslant 1} \prod_{r=1}^{n} \frac{q_r}{p_r} = \infty\,.$$

Prove that this chain is positive-recurrent if and only if

$$\sum_{n \geqslant 1} \prod_{r=1}^{n} \frac{p_{r-1}}{q_r} < \infty\,.$$

3/I/9H Markov Chains

What does it mean to say that a Markov chain is recurrent?

Stating clearly any general results to which you appeal, prove that the symmetric simple random walk on $\mathbb{Z}$ is recurrent.

4/I/9H **Markov Chains**

A Markov chain on the state–space $I = \{1, 2, 3, 4, 5, 6, 7\}$ has transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/4 & 0 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/2 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

Classify the chain into its communicating classes, deciding for each what the period is, and whether the class is recurrent.

For each $i, j \in I$ say whether the limit $\lim_{n\to\infty} p_{ij}^{(n)}$ exists, and evaluate the limit when it does exist.

1/II/19C Markov Chains

Consider a Markov chain $(X_n)_{n\geqslant 0}$ on states $\{0,1,\ldots,r\}$ with transition matrix (P_{ij}), where $P_{0,0}=1=P_{r,r}$, so that 0 and r are absorbing states. Let

$$A=(X_n=0,\ \text{for some } n\geqslant 0)\ ,$$

be the event that the chain is absorbed in 0. Assume that $h_i=\mathbb{P}(A \mid X_0=i)>0$ for $1\leqslant i<r$.

Show carefully that, conditional on the event A, $(X_n)_{n\geqslant 0}$ is a Markov chain and determine its transition matrix.

Now consider the case where $P_{i,i+1}=\frac{1}{2}=P_{i,i-1}$, for $1\leqslant i<r$. Suppose that $X_0=i$, $1\leqslant i<r$, and that the event A occurs; calculate the expected number of transitions until the chain is first in the state 0.

2/II/20C Markov Chains

Consider a Markov chain with state space $S=\{0,1,2,\ldots\}$ and transition matrix given by

$$P_{i,j}=\begin{cases} qp^{j-i+1} & \text{for } i\geqslant 1 \text{ and } j\geqslant i-1,\\ qp^{j} & \text{for } i=0 \text{ and } j\geqslant 0,\end{cases}$$

and $P_{i,j}=0$ otherwise, where $0<p=1-q<1$.

For each value of p, $0<p<1$, determine whether the chain is transient, null recurrent or positive recurrent, and in the last case find the invariant distribution.

3/I/9C Markov Chains

Consider a Markov chain $(X_n)_{n\geqslant 0}$ with state space $S=\{0,1\}$ and transition matrix

$$P=\begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}\ ,$$

where $0<\alpha<1$ and $0<\beta<1$.

Calculate $\mathbb{P}(X_n=0 \mid X_0=0)$ for each $n\geqslant 0$.

4/I/9C Markov Chains

For a Markov chain with state space S, define what is meant by the following:

(i) states $i,j\in S$ *communicate*;

(ii) state $i\in S$ is *recurrent*.

Prove that communication is an equivalence relation on S and that if two states i,j communicate and i is recurrent then j is recurrent.

1/II/19C Markov Chains

Explain what is meant by a stopping time of a Markov chain $(X_n)_{n\geq 0}$. State the strong Markov property.

Show that, for any state i, the probability, starting from i, that $(X_n)_{n\geq 0}$ makes infinitely many visits to i can take only the values 0 or 1.

Show moreover that, if

$$\sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \infty,$$

then $(X_n)_{n\geq 0}$ makes infinitely many visits to i with probability 1.

2/II/20C Markov Chains

Consider the Markov chain $(X_n)_{n\geq 0}$ on the integers $\mathbb{Z}$ whose non-zero transition probabilities are given by $p_{0,1} = p_{0,-1} = 1/2$ and

$$p_{n,n-1} = 1/3, \quad p_{n,n+1} = 2/3, \qquad \text{for } n \geq 1,$$

$$p_{n,n-1} = 3/4, \quad p_{n,n+1} = 1/4, \qquad \text{for } n \leqslant -1.$$

(a) Show that, if $X_0 = 1$, then $(X_n)_{n\geq 0}$ hits 0 with probability 1/2.

(b) Suppose now that $X_0 = 0$. Show that, with probability 1, as $n \to \infty$ either $X_n \to \infty$ or $X_n \to -\infty$.

(c) In the case $X_0 = 0$ compute $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$.

3/I/9C Markov Chains

A hungry student always chooses one of three places to get his lunch, basing his choice for one day on his gastronomic experience the day before. He sometimes tries a sandwich from Natasha's Patisserie: with probability 1/2 this is delicious so he returns the next day; if the sandwich is less than delicious, he chooses with equal probability 1/4 either to eat in Hall or to cook for himself. Food in Hall leaves no strong impression, so he chooses the next day each of the options with equal probability 1/3. However, since he is a hopeless cook, he never tries his own cooking two days running, always preferring to buy a sandwich the next day. On the first day of term the student has lunch in Hall. What is the probability that 60 days later he is again having lunch in Hall?

[*Note* $0^0 = 1$.]

4/I/9C **Markov Chains**

A game of chance is played as follows. At each turn the player tosses a coin, which lands heads or tails with equal probability 1/2. The outcome determines a score for that turn, which depends also on the cumulative score so far. Write S_n for the cumulative score after n turns. In particular $S_0 = 0$. When S_n is odd, a head scores 1 but a tail scores 0. When S_n is a multiple of 4, a head scores 4 and a tail scores 1. When S_n is even but is not a multiple of 4, a head scores 2 and a tail scores 1. By considering a suitable four-state Markov chain, determine the long run proportion of turns for which S_n is a multiple of 4. State clearly any general theorems to which you appeal.

1/II/19D **Markov Chains**

Every night Lancelot and Guinevere sit down with four guests for a meal at a circular dining table. The six diners are equally spaced around the table and just before each meal two individuals are chosen at random and they exchange places from the previous night while the other four diners stay in the same places they occupied at the last meal; the choices on successive nights are made independently. On the first night Lancelot and Guinevere are seated next to each other.

Find the probability that they are seated diametrically opposite each other on the $(n+1)$th night at the round table, $n \geqslant 1$.

2/II/20D **Markov Chains**

Consider a Markov chain $(X_n)_{n \geqslant 0}$ with state space $\{0, 1, 2, \ldots\}$ and transition probabilities given by

$$P_{i,j} = pq^{i-j+1}, \quad 0 < j \leqslant i+1, \quad \text{and} \quad P_{i,0} = q^{i+1} \quad \text{for} \quad i \geqslant 0,$$

with $P_{i,j} = 0$, otherwise, where $0 < p < 1$ and $q = 1 - p$.

For each $i \geqslant 1$, let

$$h_i = \mathbb{P}\left(X_n = 0, \text{for some } n \geqslant 0 \mid X_0 = i\right),$$

that is, the probability that the chain ever hits the state 0 given that it starts in state i. Write down the equations satisfied by the probabilities $\{h_i,\, i \geqslant 1\}$ and hence, or otherwise, show that they satisfy a second-order recurrence relation with constant coefficients. Calculate h_i for each $i \geqslant 1$.

Determine for each value of p, $0 < p < 1$, whether the chain is transient, null recurrent or positive recurrent and in the last case calculate the stationary distribution.

[*Hint: When the chain is positive recurrent, the stationary distribution is geometric.*]

3/I/9D **Markov Chains**

Prove that if two states of a Markov chain communicate then they have the same period.

Consider a Markov chain with state space $\{1, 2, \ldots, 7\}$ and transition probabilities determined by the matrix

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Identify the communicating classes of the chain and for each class state whether it is open or closed and determine its period.

4/I/9D **Markov Chains**

Prove that the simple symmetric random walk in three dimensions is transient.

[*You may wish to recall Stirling's formula:* $n! \sim (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n}$.]

1/I/11H Markov Chains

Let $P = (P_{ij})$ be a transition matrix. What does it mean to say that P is (a) irreducible, (b) recurrent?

Suppose that P is irreducible and recurrent and that the state space contains at least two states. Define a new transition matrix $\tilde{P}$ by

$$\tilde{P}_{ij} = \begin{cases} 0 & \text{if} \quad i = j, \\ (1 - P_{ii})^{-1} P_{ij} & \text{if} \quad i \neq j. \end{cases}$$

Prove that $\tilde{P}$ is also irreducible and recurrent.

1/II/22H Markov Chains

Consider the Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Determine the communicating classes of the chain, and for each class indicate whether it is open or closed.

Suppose that the chain starts in state 2; determine the probability that it ever reaches state 6.

Suppose that the chain starts in state 3; determine the probability that it is in state 6 after exactly n transitions, $n \geqslant 1$.

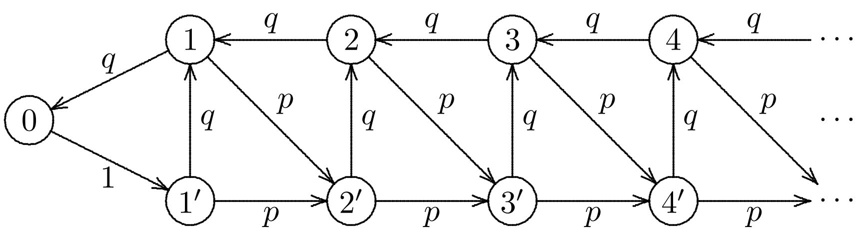
2/I/11H Markov Chains

Let $(X_r)_{r \geqslant 0}$ be an irreducible, positive-recurrent Markov chain on the state space S with transition matrix (P_{ij}) and initial distribution $P(X_0 = i) = \pi_i$, $i \in S$, where (π_i) is the unique invariant distribution. What does it mean to say that the Markov chain is reversible?

Prove that the Markov chain is reversible if and only if $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$.

2/II/22H **Markov Chains**

Consider a Markov chain on the state space $S = \{0, 1, 2, \ldots\} \cup \{1', 2', 3', \ldots\}$ with transition probabilities as illustrated in the diagram below, where $0 < q < 1$ and $p = 1 - q$.



For each value of q, $0 < q < 1$, determine whether the chain is transient, null recurrent or positive recurrent.

When the chain is positive recurrent, calculate the invariant distribution.