

1. Suppose X_1, \dots, X_n are independent copies of a random variable X taking values in some interval $[a, b]$. Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the order statistics of the data (assume for simplicity that there are no ties almost surely). The sample median $\hat{\theta}$ is given by

$$\hat{\theta} = \begin{cases} X_{((n+1)/2)} & \text{if } n \text{ is odd,} \\ \frac{1}{2}(X_{(n/2)} + X_{(n/2+1)}) & \text{if } n \text{ is even.} \end{cases}$$

- (a) Show that $\hat{\theta}$ minimises

$$M_n(\theta) := \frac{1}{n} \sum_{i=1}^n |X_i - \theta|$$

over $\theta \in [a, b]$.

- (b) Suppose θ_0 is such that $\mathbb{P}(X \leq \theta_0) = 1/2$ and assume additionally that for all $\epsilon > 0$, we have

$$\mathbb{P}(X \leq \theta_0 - \epsilon) < 1/2 < \mathbb{P}(X \leq \theta_0 + \epsilon).$$

Show that θ_0 is the unique minimiser over θ of $\theta \mapsto \mathbb{E}|X - \theta|$. [*Hint: First aim to show that when $\theta > \theta_0$, $\mathbb{E}|X - \theta| - \mathbb{E}|X - \theta_0| = 2\mathbb{E}\{\mathbb{1}_{\{\theta_0 < X \leq \theta\}}(\theta - X)\}$.]*

- (c) Show that $\hat{\theta} \xrightarrow{P} \theta_0$. [*Hint: Apply a (version of) a result from lectures.*]

2. Let $\Theta \subseteq \mathbb{R}$ have non-empty interior, and let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of real-valued continuous functions defined on Θ such that $\psi_n(\theta) \xrightarrow{P} \psi(\theta)$ for all $\theta \in \Theta$, where ψ is some deterministic function $\psi : \Theta \rightarrow \mathbb{R}$. Suppose that for some $\theta_0 \in \text{int } \Theta$ and every $\delta > 0$ sufficiently small, we have $\psi(\theta_0 \pm \delta) < 0 < \psi(\theta_0 \mp \delta)$. Suppose moreover that ψ_n has exactly one zero $\hat{\theta}_n \in \Theta$ for every $n \in \mathbb{N}$. Deduce that $\hat{\theta}_n \xrightarrow{P} \theta_0$ as $n \rightarrow \infty$.

3. Let X_1, \dots, X_n be i.i.d. observations from the model

$$\{f(x, \theta) = \theta x^{\theta-1} \exp(-x^\theta) \mathbb{1}_{\{x>0\}} : \theta \in (0, \infty)\}$$

of Weibull distributions. Show that the MLE exists almost surely and is consistent. [*Hint: Use the previous question. You may interchange differentiation and integration without proof in your answer.*]

4. Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} N(\theta, 1)$ where $\theta \in \Theta = [0, \infty)$. Find the MLE $\hat{\theta}$. Show that when the true parameter $\theta_0 > 0$ then $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, 1)$. What happens when $\theta_0 = 0$? Comment on your findings in the light of the general asymptotic theory of MLEs.

5. Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} U[0, \theta]$ with $\theta \in \Theta = (0, \infty)$.

- (a) Find the MLE $\hat{\theta}$ and show that $\tilde{\theta} = (n+1)\hat{\theta}/n$ is unbiased for estimating θ .
 (b) Find the variance of $\tilde{\theta}$ and comment on this in relation to the form of the Cramér–Rao lower bound in regular models.
 (c) Finally, find the asymptotic distribution of $n(\hat{\theta} - \theta)$.

6. Consider the setup of Question 5 from Example Sheet 1 where $(Y_1, X_1), \dots, (Y_n, X_n) \stackrel{\text{i.i.d.}}{\sim} N_2(\mu, \Sigma)$ where Σ is known and $\mu \in \mathbb{R}^2$. Show that the Wald test, score test and generalised likelihood ratio test for testing the composite null hypothesis $H_0 : \mu_2 = \mu_0$ are identical and deliver exact (non-asymptotic) size control.
7. Suppose $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$ where $\theta \in (0, 1)$.
- (a) Write down a $(1-\alpha)$ -level Wald confidence interval for θ (estimating $I_1(\theta_0)$, where θ_0 is the true parameter, by $I_1(\bar{X})$). What happens when $\min(\bar{X}, (1-\bar{X})) < z_\alpha^2(n+z_\alpha^2)^{-1}$, where z_α is the upper $\alpha/2$ point of a standard Gaussian?
- (b) Now let $\phi(u) := \log(u/(1-u))$ for $u \in (0, 1)$. By considering the asymptotic distribution of $\sqrt{n}(\phi(\bar{X}) - \phi(\theta))$, find a different confidence interval for θ . Briefly comment on why this second approach might be preferred.
8. We revisit Question 3 from Example Sheet 1. Suppose we have n i.i.d. copies X_1, \dots, X_n of $X^{(1)} \sim N(\mu_1, 1)$ and also n i.i.d. observations W_1, \dots, W_n , independent of X_1, \dots, X_n , each having distribution equal to that of $\text{sgn}(X^{(2)})$ where $X^{(2)} \sim N(\mu_2, 1)$. Without using the general asymptotic theory concerning MLEs, find an asymptotic $(1-\alpha)$ -level confidence interval for $\mu_0 := \mu_1 - \mu_2$.
9. Consider the model $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$ where $\theta > 0$ (so the density is $\theta e^{-\theta x}$). We wish to test the null hypothesis $\theta = \theta_0$. Recall that the power of a test $\phi_n = \phi_n(X_1, \dots, X_n)$ where $\phi_n = 1$ indicates rejection (and 0 acceptance) at the alternative $\theta_1 \neq \theta_0$ is $\mathbb{P}_{\theta_1}(\phi_n = 1)$. Consider a sequence $\theta_n = \theta_0 + h/\sqrt{n}$ (where $h \neq 0$ is some fixed real number) of *local alternatives*. Show that the limit

$$\lim_{n \rightarrow \infty} \mathbb{P}_{\theta_n}(\phi_n = 1)$$

is the same for the score and Wald tests with asymptotic size α and find this limit. [*Hint: First argue that if $X \sim \text{Exp}(\theta)$ then $a^{-1}X \sim \text{Exp}(a\theta)$ for $a > 0$.]*

10. Let $\sigma^2 > 0$ be deterministic and suppose $X_1, \dots, X_n | \theta \stackrel{\text{i.i.d.}}{\sim} N(\theta, \sigma^2)$ with prior distribution $\theta \sim N(\mu, v^2)$, where $\mu \in \mathbb{R}$ and $v^2 > 0$. Show that the posterior distribution of θ is given by

$$\theta | X_1, \dots, X_n \sim N\left(\frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu}{v^2}}{\frac{n}{\sigma^2} + \frac{1}{v^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{v^2}}\right)$$

11. Consider $X_1, \dots, X_n | \mu, \sigma^2 \stackrel{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$ with *improper prior* density $\pi(\mu, \sigma)$ proportional to σ^{-2} (constant in μ). Argue that the resulting ‘posterior distribution’ has a density proportional to

$$\sigma^{-(n+2)} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu)^2\right\},$$

and that thus the distribution of $\mu | \sigma^2, X_1, \dots, X_n$ is $N(\bar{X}, \sigma^2/n)$. Assuming σ^2 is known, for $\alpha \in (0, 1)$ construct a $(1-\alpha)$ -level credible set for the posterior distribution $\mu | \sigma^2, X_1, \dots, X_n$ that is also an exact $(1-\alpha)$ -level (frequentist) confidence set.